BIORTHOGONAL SYSTEMS OF FUNCTIONS*

BY

ANNA JOHNSON PELL

Introduction.

In boundary value problems of differential equations which are not self-adjoint, biorthogonal systems of functions play the same rôle as the orthogonal systems do in the self-adjoint case. LIOUVILLE† has considered special non-self-adjoint differential equations with real characteristic values of the parameter; BIRKHOFF‡ has proved the existence of the characteristic values (in general complex) for the differential equation of the nth order and obtained the related expansions.

If the integral equation

$$u(s) = \lambda \int_a^b L(s, t)u(t)dt$$

with the unsymmetric kernel $L(s, t)$ has solutions $u(s)$, and therefore the integral equation

$$v(s) = \lambda \int_a^b L(t, s)v(t)dt$$

solutions $v(s)$, it has been shown by PLEMELJ§ and GOURSAT‖ that the solutions or functions closely related to them form a biorthogonal system. But expansions in terms of these solutions have not been obtained, and no criteria have been given for the existence of real characteristic numbers of an unsymmetric kernel.¶

The object of this paper is the development of a theory of biorthogonal systems of functions independent of their connection with integral or differential equations. In the theory frequent use is made of the theorems by RIESZ, FISCHER, and TOEPLITZ (Lemmas 1, 2, 3, and 4, § 2).

* Presented to the Society (Chicago) April 10, 1909.

† Liouville's Journal, ser. 1, vol. 3 (1838).

‡ Transactions of the American Mathematical Society, vol. 9 (1908).

§ Monatshefte für Mathematik und Physik, vol. 15 (1904).

‖ Annales de Toulouse, 1908.

¶ Since this paper was written J. MARTY has published some results for unsymmetric kernels, Comptes Rendus, February 28, March 7, April 25, and June 6, 1910. See also the note by the author in Bulletin of the American Mathematical Society, July, 1910.

Necessary and sufficient conditions for the existence of the adjoint system $\{v_i\}$ of any system of linearly independent functions $\{u_i\}$ are deduced (theorem 2, § 3). Theorems for biorthogonal systems analogous to those of RIESZ and FISCHER for orthogonal systems are: (*a*) if

$$\sum_{i=1}^{\infty}\left(\int fv_i\right)^2$$

converges for every function f which is integrable and has an integrable square, then for every system of constants $\{c_i\}$ of finite norm there exists a function g such that $c_i = \int gu_i$ (corollary 1, theorem 6); (*b*) if the system $\{u_i\}$ is complete and if certain conditions are imposed on the function g, then (theorem 8)

$$\int fg = \sum_{i=1}^{\infty} \int fv_i \int u_i g.$$

The equivalence of two biorthogonal systems is defined and a classification into types is made (§ 7). With each type satisfying a certain condition there is connected uniquely (§ 6) and in a reversible way (theorem 20, § 8) a single-valued functional transformation* $T(f)$, which transforms every function which is integrable and has an integrable square into a function of the same kind, and is defined by the properties

(1) $$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2),$$

(2) $$\int [f_1 T(f_2) - f_2 T(f_1)] = 0,$$

(3) $$\int f T(f) \geqq 0.$$

The class of all orthogonal systems of functions is a special type, for which $T(f)$ is the identical transformation. Other special cases of $T(f)$ are

$$T(f) = f(s) - \frac{p(s)\int_a^b p(s) f(s)\, ds}{\int_a^b [p(s)]^2 ds},$$

$$T(f) = \int_a^b K(s, t) f(t)\, dt.$$

By means of the theorems by RIESZ, FISCHER and TOEPLITZ, it can be shown that there is a one-to-one correspondence between this functional transformation $T(f)$ and positive definite limited quadratic forms in infinitely many variables.

*Linear functional transformations have recently been studied by F. RIESZ, Mathematische Annalen, vol. 69 (1910), p. 449.

§ 1. *Fundamental Notations.*

We consider an arbitrary interval I: $a \leqq s \leqq b$: of the real variable s, and the corresponding square S: $a \leqq s \leqq b$, $a \leqq t \leqq b$: of the real variables s and t, and real functions of these variables. We denote functions of a single variable by f, g, φ, ϕ, etc., omitting the argument. $\mathfrak{F}$ is the class of all functions f such that f and f^2 are integrable in the sense of Lebesgue* on the interval I. As usual, we regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero.† We call two such functions "essentially equal."

When there is no ambiguity we omit the variable of integration In all cases the limits of integration are omitted, since I is the only interval of integration considered in this paper.

Constants and functions with the subscripts i, j, k denote sequences of constants and functions.

$\mathfrak{G}$ and $\mathfrak{H}$ denote subclasses of $\mathfrak{F}$ and the elements of these subclasses are denoted by the small letters g and h.

To express that a relation involving the function g holds for every function of the class we write (g) after the relation; for example

$$\int f^2 \geqq 0 \qquad (f).$$

We use a similar notation in the case of several variables; for example

$$\left(\int f_1 f_2\right)^2 \leqq \int f_1^2 \int f_2^2 \qquad (f_1, f_2).$$

Let $\{f_i\}$ denote any sequence of linearly independent functions of the class $\mathfrak{F}$, and $\{\phi_i\}$ the sequence of normalized and orthogonal‡ functions which are obtained linearly from the functions f_i by means of the construction given by E. SCHMIDT,§

$$\phi_i = \frac{f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i\right)^2}} \qquad (i).$$

Each ϕ_i may be expressed linearly and homogeneously with constant coefficients in terms of $f_1, f_2, \cdots, f_i$, and conversely:

$$(1) \qquad \phi_i = \sum_{k=1}^{i} a_{ik} f_k, \qquad f_i = \sum_{k=1}^{i} b_{ik} \phi_k \qquad (i).$$

* LEBESGUE, *Sur l'intégration*, p. 115.

† LEBESGUE, l. c., p. 106.

‡ A function f is normalized if $\int f^2 = 1$. Two functions f_1 and f_2 are orthogonal if $\int f_1 f_2 = 0$.

§ *Zur Theorie der linearen und nichtlinearen Integralgleichungen.* Mathematische Annalen, vol. 63 (1907), p. 442.

The constants a_{ik}, b_{ik} have the following properties:

$$
\begin{aligned}
&a_{ii} \neq 0 && (i),\\
&a_{ik} = 0 && (k>i),\\
&b_{ii} \neq 0 && (i),\\
&b_{ik} = 0 && (k>i).
\end{aligned}
\tag{2}
$$

The matrices formed from the coefficients a_{ik} and b_{ik} we denote by $(a_{ik})_f$ and $(b_{ik})_f$ respectively, or simply by $(a)_f$ and $(b)_f$.

If $\{\bar{\phi}_i\}$ are the normalized and orthogonal functions corresponding to $\{\bar{f}_i = \lambda_i f_i\}$ where $\{\lambda_i\}$ are constants different from zero, then $\bar{\phi}_i = \phi_i$.

§ 2. *Preliminary definitions and lemmas.*

Definition. A system of functions of $\mathfrak{F}$ is called *complete* * for the interval I if there exists no function f of $\mathfrak{F}$, essentially different from zero, which is orthogonal to all the functions of the system in the interval I.

Definition. A sequence of constants $\{c_i\}$ is of *finite norm* if the sum of the squares,

$$\sum_{i=1}^{\infty} c_i^2,$$

converges, and it has the norm M if

$$\sum_{i=1}^{\infty} c_i^2 = M.$$

For any system $\{\phi_i\}$ of normalized and orthogonal functions of $\mathfrak{F}$ we have the three following lemmas.†

Lemma 1. The sequence of constants

$$\left\{\int f\phi_i\right\}$$

is of finite norm for every function f of $\mathfrak{F}$.

Lemma 2. If the system $\{\phi_i\}$ is complete,

$$\int f_1 f_2 = \sum_{i=1}^{\infty} \int f_1 \phi_i \int \phi_i f_2 \qquad (f_1, f_2).$$

* F. RIESZ, Comptes Rendus, November, 1906, p. 738.

† E. SCHMIDT, l. c., p. 439; F. RIESZ, Comptes Rendus, March, 1907, p. 615, and April, 1907, p. 734; E. FISCHER, Comptes Rendus, May, 1907, p. 1022.

Lemma 3. To every system of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f\phi_i \qquad (i),$$

and this function f is essentially unique if the system $\{\phi_i\}$ is complete.

Definition. A matrix (α_{ik}) is *limited* if

$$\sum_{i=1}^{\infty} \alpha_{ik} x_i$$

converges for every k, and

$$\left\{ \sum_{i=1}^{\infty} \alpha_{ik} x_i \right\}$$

is of finite norm for every system of values of $\{x_i\}$ of finite norm.

Definition. A bilinear form of the infinitely many variables x_i and y_i

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{ik} x_i y_k$$

is limited if for some positive constant M

$$\left| \sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{ik} x_i y_k \right| \leqq M \qquad (n)$$

for all values of $\{x_i\}$ and $\{y_i\}$ of norm $\leqq 1$ (H, IV, p. 176 *).

Lemma 4. The matrix of the coefficients of a limited bilinear form is limited (H, IV, p. 179); conversely, in virtue of a theorem by TOEPLITZ † a bilinear form whose coefficients are the elements of a limited matrix is limited.

Lemma 5. If the matrix (α_{ik}) is limited the matrix (α_{ki}) is limited also.

Lemma 6. The product of two limited matrices is a limited matrix (H, IV, p. 179).

Definition. A matrix (α_{ik}) is orthogonal if

$$\sum_{i=1}^{\infty} \alpha_{ij} \alpha_{ik} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad (j, k),$$

$$\sum_{i=1}^{\infty} \alpha_{ji} \alpha_{ki} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad (j, k).$$

Lemma 7. An orthogonal matrix is limited (H, IV, p. 180). If some of the elements of the orthogonal matrix are replaced by zeros, the resulting matrix is limited.

* The references to the memoirs by HILBERT in the Göttinger Nachrichten, 1904–1906, are denoted by (H, I, II, ···, V).

† E. SCHMIDT, Palermo Rendiconti, vol. 25 (1908), p. 2; HELLINGER and TOEPLITZ, Göttinger Nachrichten, 1906, p. 351, and Mathematische Annalen, vol. 69 (1910), p. 289.

Lemma 8. The value of a limited bilinear form for arguments $\{x_i\}$ and $\{y_i\}$ of norm 1 is obtained by summation either by rows or by columns (*H*, IV, p. 180).

Lemma 9. If α_{ik} are the elements of a limited matrix, and if $\alpha_{ik} = \alpha_{ki}$, then the system of homogeneous linear equations

$$x_i - \lambda \sum_{k=1}^{\infty} \alpha_{ik} x_k = 0 \qquad (i)$$

has a solution $\{x_i\}$ of finite norm when and only when λ is a "characteristic number," that is, a root of a certain associated equation, and the number of linearly independent solutions is equal to the multiplicity of the root (*H*, IV, p. 198).

Definition. Two systems* (u_i) and (v_i) of functions of $\mathfrak{F}$ form a *biorthogonal system* of functions (u_i, v_i) if a one-to-one correspondence can be established between them such that the integral of the product of two corresponding functions is equal to unity and the integral of the product of two non-corresponding functions is equal to zero; i. e.,

$$\int u_i v_j = \begin{cases} 1 & (i=j), \\ 0 & (i \neq j). \end{cases} \qquad (3)$$

Each one of the two systems (u_i), (v_i) is called an *adjoint* system of the other.

The biorthogonal system (u_i, v_i) is *complete as to* u, v, respectively, if the system (u_i), (v_i), respectively, is complete.

For a biorthogonal system (u_i, v_i) the functions u_i and also v_i are linearly independent. Suppose there existed a linear relation

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0.$$

Multiply by $v_i (i = 1, 2, \cdots, n)$ and integrate; from the properties (3) we obtain

$$c_i = 0 \qquad (i = 1, 2, \cdots, n).$$

§ 3. *Denumerability and existence of the adjoint system.*

Theorem 1. *A system (u_i) of functions of $\mathfrak{F}$ having an adjoint system (v_i) is denumerable.*

We follow the method used by RIESZ † to prove that an orthogonal system of functions is denumerable. The proof is based on the following theorem: If for

* Until after theorem 1 the subscripts i and j are used to denote not only sequences but any system of functions.

† Comptes Rendus, November, 1906, p. 738.

every sequence $\{f_i\}$ of linearly independent functions selected from a given system

$$\int (f_{i_1} - f_{i_2})^2 \geqq 1 \qquad (i_1, i_2),$$

then the given system is denumerable.

Without loss of generality we may assume that

$$\int v_i^2 = 1 \qquad (i).$$

Let $\{u_i\}$ be any sequence of distinct functions selected from the system (u_i) and let v_i be the corresponding functions of the adjoint system. By Bessel's inequality we have

$$1 = \Big[\int v_{i_1}(u_{i_1} - u_{i_2})\Big]^2 \leqq \int (u_{i_1} - u_{i_2})^2 \qquad (i_1, i_2).$$

The hypotheses of the theorem are satisfied and the system (u_i) is denumerable. When the system (u_i) is complete it cannot be finite, and is therefore denumerably infinite.

Let $\{\varphi_i\}$ and $\{\psi_i\}$ be the systems of normalized orthogonal functions constructed linearly from the functions $\{u_i\}$ and $\{v_i\}$, respectively, of the biorthogonal system (u_i, v_i). Connected with these systems of functions we have the four matrices $(a)_u$, $(b)_u$, $(a)_v$, $(b)_v$. (§ 1.)

Theorem 2. *The necessary and sufficient condition that a given system of functions* $\{u_i\}$ *of* $\mathfrak{F}$ *have an adjoint system is, that the* a_{ik} *of* $(a)_u$ *be of finite norm for every* k.

Suppose the condition is satisfied; then from lemma 3 follows the existence of a system of functions $\{v_i\}$ of $\mathfrak{F}$ such that

$$a_{ik} = \int \varphi_i v_k \qquad (i, k). \tag{4}$$

The equations connecting $\{u_i\}$ and $\{\varphi_i\}$ are

$$\varphi_i = \sum_{k=1}^{i} a_{ik} u_k \qquad (i) \tag{5}$$

and

$$u_i = \sum_{k=1}^{i} b_{ik} \varphi_k. \tag{6}$$

Multiply the equation (6) by v_j and integrate:

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} \int \varphi_k v_j.$$

By substitution from the equation (4):

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} a_{kj}.$$

Since the matrices (b_{ik}) and (a_{kj}) are each the inverse of the other we obtain

$$\int u_i v_j = \begin{cases} 0 & i = j, \\ 1 & i \neq j. \end{cases}$$

Thus by definition the system $\{v_i\}$ is an adjoint system of $\{u_i\}$.

Suppose conversely that an adjoint system $\{v_i\}$ exists. Multiply the equations (5) by v_k and integrate

$$a_{ik} = \int \varphi_i v_k \qquad (i, k).$$

Applying lemma 1, we see that the condition is necessary.

Example. Let $u_i = s^{i-1}$, $a = -1$, $b = +1$. Since the Legendrian polynomials $P_i(s)$ form an orthogonal system of functions for the interval $(-1, +1)$, and since $P_i(s)$ is linear in $1, s, s^2, \cdots, s^i$, it is clear that the functions φ_i defined by the equations (5) are the Legendrian polynomials multiplied by the proper constants. The value* of $a_{2m,1}$ is given by

$$a_{2m,1} = \sqrt{\frac{4m+1}{2}}\,\frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdots 2m}.$$

Hence

$$a^2_{2m,1} > \frac{4m+1}{2(2m+1)} \cdot \frac{2}{\pi}.$$

And the constants $\{a_{2m,1}\}$ cannot be of finite norm. This is sufficient to show that the system of the powers of s has no adjoint system in the interval $(-1, +1)$.

§ 4. *Limited matrices.*

Lemma 1, § 2, can be extended to biorthogonal systems only when certain conditions indicated in the following theorem are satisfied.

Theorem 3. *If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{\int f f_i\right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that the matrix $(b)_j$ be limited.*

Multiply the second equation of (1) by any function f of $\mathfrak{F}$ and integrate:

$$\int f f_i = \sum_{k=1}^{i} b_{ik} \int \phi_k f \qquad (i, f).$$

By lemma 1, $\left\{\int \phi_k f\right\}$ is of finite norm for every f of $\mathfrak{F}$, and therefore $\left\{\int f f_i\right\}$ is of finite norm for every f of $\mathfrak{F}$ if the matrix $(b)_j$ is limited.

On the other hand, if

$$\left\{\int f f_i\right\}$$

* BYERLY, *Fourier's Series*, pp. 180 and 70.

is of finite norm for every f, then

$$\left\{\sum_{k=1}^{i} b_{ik}x_k\right\}$$

is of finite norm for every $\{x_i\}$ of finite norm because

$$\int ff_i = \sum_{k=1}^{i} b_{ik}x_k,$$

where f is the function of $\mathfrak{F}$ which by lemma 3 corresponds to $\{x_i\}$.

Remark. We can always multiply the functions $\{f_i\}$ by constants $\lambda_i \neq 0$ so that

$$\left\{\lambda_i \int ff_i\right\} \qquad (f)$$

is of finite norm. This is evident since it follows from Bessel's inequality that

$$\sum_{i=1}^{\infty}\left(\lambda_i \int ff_i\right)^2 \leqq \int f^2 \sum_{i=1}^{\infty}\lambda_i^2 \int f_i^2 \qquad (f).$$

Corollary 1. If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{\int ff_i\right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that

$$\Sigma\left(\int ff_i\right)^2 \leqq M\int f^2,$$

where M is a constant depending only on the sequence $\{f_i\}$.

This is a consequence of the property of a limited matrix that

$$\sum_i\left(\sum_k a_{ik}x_k\right)^2 \leqq M\Sigma x_i^2,$$

where M does not depend on Σx_i^2 (lemma 4).

Corollary 2. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system of functions, the necessary and sufficient condition that $\left\{\int fu_i\right\}$, $\left\{\int fv_i\right\}$, respectively, be of finite norm $\leqq M\int f^2$ for every function f of $\mathfrak{F}$ is that $(b)_u$, $(b)_v$, respectively, be limited.

Theorem 4. *If the biorthogonal system (u_i, v_i) is complete as to u, v, respectively, the necessary and sufficient condition that $\left\{\int fv_i\right\}$, $\left\{\int fu_i\right\}$, respectively, be of finite norm for every function f of $\mathfrak{F}$ is that the matrices $(a)_u$, $(a)_v$, respectively, be limited.*

We prove the theorem for the case that $\{u_i\}$ is complete. From the equations (5) it is clear that the system $\{\varphi_i\}$ is also complete. Applying lemma 2 and the equations (4) we obtain the following equations:

$$\int fv_i = \sum_{k=1}^{\infty} a_{ki}\int \varphi_k f \qquad (i, f). \tag{7}$$

The remainder of the proof is precisely the same as for theorem 3.

Corollary. When the biorthogonal system (u_i, v_i) is complete as to u, v, respectively, a limited matrix $(a)_u$, $(a)_v$, respectively, implies a limited matrix $(b)_v$, $(b)_u$, respectively, and conversely.

Theorem 5. *If* $\{f_i\}$ *is any sequence of linearly independent functions of* $\mathfrak{F}$, *the necessary and sufficient condition, that for every sequence of constants* $\{c_i\}$ *of finite norm there may exist a function* f *of* $\mathfrak{F}$ *such that*

$$c_i = \int f f_i \qquad (i),$$

is that the matrix $(a)_f$ *be limited.*

Multiply the first set of equations (1) by an arbitrary function f of $\mathfrak{F}$ and integrate:

$$\int f\phi_i = \sum_{k=1}^{i} a_{ik} \int f_k f \qquad (i). \tag{8}$$

Suppose first that the matrix $(a)_f$ is limited; then the sequence of constants

$$\left\{ \sum_{k=1}^{i} a_{ik} c_k \right\}$$

is of finite norm and there exists [lemma 3] a function f of $\mathfrak{F}$ such that

$$\int f \phi_i = \sum_{k=1}^{i} a_{ik} c_k \qquad (i).$$

Taking into consideration the properties (2), § 1, it is clear from the equations (8) that

$$c_i = \int f_i f \qquad (i).$$

We now make the assumption that to every sequence of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f_i f \qquad (i).$$

By means of the equations (8) and lemma 1 we find that

$$\sum_{k=1}^{i} a_{ik} c_k$$

is of finite norm for every sequence $\{c_i\}$ of finite norm, and thus by definition the matrix $(a)_f$ is limited.

If the system $\{f_i\}$ is complete, then

$$\int f^2 \leqq M \sum_i c_i^2.$$

Corollary. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system, the necessary and sufficient condition, that for every sequence of constants $\{c_i\}$ of finite norm there exists a function f of $\mathfrak{F}$ such that for every i, $c_i = \int u_i f$, $c_i = \int v_i f$, respectively, is that the matrix $(a)_u$, $(a)_v$, respectively, be limited.

Theorem 6. *If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system of functions, a limited matrix $(b)_v$, $(b)_u$, respectively, implies a limited matrix $(a)_u$, $(a)_v$, respectively.*

We give the proof for a limited matrix $(b)_v$. The matrix $(b)_v$ is formed from the coefficients in the equations

$$v_i = \sum_{j=1}^{i} b_{ij} \psi_j \qquad (i).$$

Multiply these equations by φ_k and integrate:

$$a_{ki} = \sum_{j=1}^{i} b_{ij} \int \psi_j \varphi_k \qquad (i, k), \tag{9}$$

where by equations (4) a_{ki} are the elements of the matrix $(a_{ki})_u$. The matrix $(a_{ki})_u$ is the product of the two matrices $(b_{ij})_v$ and $\left(\int \psi_j \varphi_k\right)$ which are both limited (lemma 7), and is therefore (lemma 6) itself limited.

For a biorthogonal system (u_i, v_i), the functions u_i can be multiplied by constants $\lambda_i \neq 0$ so that the resulting matrix $(a)_{\lambda u}$ is limited.

Corollary 1. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system such that the sequence $\left\{\int f v_i\right\}$ is of finite norm for every f of $\mathfrak{F}$, then for every sequence of constants $\{c_i\}$ of finite norm there exists a function g of $\mathfrak{F}$ such that

$$c_i = \int g u_i.$$

Corollary 2. When the biorthogonal system (u_i, v_i) is complete as to u, the elements of the matrices $(a)_u$ and $(b)_v$ are connected by the relations

$$\sum_k a_{ki}^2 = \sum_k b_{ik}^2 \qquad (i). \tag{10}$$

A similar relation exists between the elements of the matrices $(a)_v$ and $(b)_u$ when the biorthogonal system is complete as to v.

We give below four examples to illustrate the different cases which may arise.

Example 1. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I, and $\{a_i\}$ a system of constants such that $|a_i| \leqq M$ where M is some finite positive quantity. Then $\{u_i\}$ and $\{v_i\}$, defined by

$$u_{2j-1} = \phi_{2j-1}, \qquad u_{2j} = \phi_{2j} + a_j \phi_{2j-1}, \qquad v_{2j-1} = \phi_{2j-1} - a_j \phi_{2j}, \qquad v_{2j} = \phi_{2j},$$

form a biorthogonal system (u_i, v_i) whose matrices $(b)_u$, $(b)_v$ and therefore $(a)_v$, $(a)_u$ are limited.

Example 2. The same biorthogonal system as in example 1, except that the constants $\{a_i\}$ are such that $\{1/a_i\}$ is of finite norm. In this case neither of the matrices $(b)_u$ and $(b)_v$ is limited.

Example 3. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I: $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm. For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = a_i\phi_i \qquad (i),$$

$$v_i = \frac{\phi_i}{a_i} \qquad (i),$$

the matrix $(b)_v$ is limited, but the matrix $(b)_u$ is not.

Example 4. The systems $\{\phi_i\}$ and $\{a_i\}$ are conditioned as in example (3). For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = \phi_{2i-1} \qquad (i),$$

$$v_i = \phi_{2i-1} + a_i\phi_{2i} \qquad (i),$$

the matrix $(a)_u$ is limited but the matrix $(b)_v$ is not. In this example the matrices $(a)_v$ and $(b)_u$ are also limited but the system $\{u_i\}$ is not complete: compare with theorem 4.

§ 5. *Convergence and evaluation of*

$$\sum_{i=1}^{\infty} \int f_1 u_i \int v_i f_2 .$$

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{G}$ is the class of all functions g for which there exists a system of constants $\{\mu_i \neq 0\}$ such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, and $\left\{\int f v_i/\mu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

Example. For the biorthogonal system given in example 2, § 4, the class $\mathfrak{G}$ is the class of all functions g such that $\left\{c_i \int g u_i\right\}$ is of finite norm, where $c_{2i-1} = a_i$, $c_{2i} = 1$. It is evident that the matrix $(b)_{v/\mu}$ is limited. Any other system of constants $\{\mu_i\}$ which would make $(b)_{v/\mu}$ limited is such that the ratio c_i/μ_i is less than a fixed constant for every i, and therefore for any function g such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, the sequence $\left\{c_j \int g u_i\right\}$ is also of finite norm.

When $(b)_v$ is limited, $\mathfrak{G}$ contains the class of all functions g of $\mathfrak{F}$ such that $\left\{\int g u_i\right\}$ is of finite norm.

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{H}$ is the class of all functions h for which there exists a system of constants $\{\nu_i \neq 0\}$ such that $\left\{\nu_i \int h v_i\right\}$ is of finite norm, and $\left\{\int f u_i/\nu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

When $(b)_u$ is limited, $\mathfrak{H}$ contains the class of all functions h of $\mathfrak{F}$ such that $\left\{\int h v_i\right\}$ is of finite norm.

In the definition the function g is assumed to belong to the class $\mathfrak{F}$, but any function g satisfying those conditions must belong to $\mathfrak{F}$ since the matrix $(b)_{v\mu}$ and therefore the matrix $(a)_{\mu\nu}$ corresponding to g is limited, and by the equations (5), $\left\{\int g\varphi_i\right\}$ is of finite norm and therefore g belongs to $\mathfrak{F}$. Similarly the functions h must necessarily belong to $\mathfrak{F}$.

Theorem 7. *If (u_i, v_i) is a biorthogonal system, then for every function g of $\mathfrak{G}$ and for every function f of $\mathfrak{F}$ the expression*

$$\sum_{i=1}^{\infty} \int g u_i \int v_i f$$

is convergent; and for every function h of $\mathfrak{H}$ and every function f of $\mathfrak{F}$ the expression

$$\sum_{i=1}^{\infty} \int f u_i \int v_i h$$

is convergent.

The convergence follows directly from the two inequalities:

$$\left(\sum_{i=1}^{\infty} \int g u_i \int v_i f\right)^2 \leqq \sum_{i=1}^{\infty} \left(\mu_i \int g u_i\right)^2 \sum_{i=1}^{\infty} \left(\frac{\int f v_i}{\mu_i}\right)^2 \qquad (f, g),$$

$$\left(\sum_{i=1}^{\infty} \int f u_i \int v_i h\right)^2 \leqq \sum_{i=1}^{\infty} \left(\frac{\int f u_i}{\nu_i}\right)^2 \sum_{i=1}^{\infty} \left(\nu_i \int h v_i\right)^2 \qquad (f, h).$$

Theorem 8. *When the biorthogonal system (u_i, v_i) is complete as to u, or when the functions v_i and g are orthogonal to the functions which together with the u_i form a complete system, the integral $\int g f$ may be represented as follows:*

$$\int g f = \sum_{i=1}^{\infty} \int g u_i \int v_i f \qquad (f, g); \tag{11}$$

when it is complete as to v, or when the functions u_i and h are orthogonal to the functions which together with the v_i form a complete system, the integral $\int h f$ may similarly be represented:

$$\int h f = \sum_{i=1}^{\infty} \int f u_i \int v_i h \qquad (f, h). \tag{12}$$

We give the proof for the first part of the theorem.

From the definition of $\mathfrak{G}$ the matrix $(b)_{\nu\mu}$ is limited, and therefore by theorem 6 the matrix $(a)_{\mu\nu}$ is limited.

The elements of the limited matrix $(a)_{\mu\nu}$ are a_{ik}/μ_k. Consider the bilinear form

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}x_k y_i.$$

Make the substitution

$$x_k = \mu_k \int g u_k \qquad (k),$$

$$y_i = \int \varphi_i f \qquad (i).$$

Since

$$\frac{\int f v_k}{\mu_k} = \sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}\int \varphi_i f \qquad (k)$$

we obtain from summation by rows

$$\sum_{k=1}^{\infty}\int g u_k \int v_k f.$$

Since

$$\int f\varphi_i = \sum_{k=1}^{i}\frac{a_{ik}}{\mu_k}\mu_k \int u_k f$$

we obtain from summation by columns

$$\sum_{i=1}^{\infty}\int g\varphi_i \int \varphi_i f.$$

By lemma 7 these two sums are equal, and applying lemma 2 we have the desired result:

$$\int gf = \sum_{i=1}^{\infty}\int g u_i \int v_i f \qquad (f, g).$$

Corollary 1. When the biorthogonal system (u_i, v_i) is complete as to u, then

$$\int g^2 = \sum_{i-1}^{\infty}\int g u_i \int v_i g \qquad (g) \tag{13}$$

when it is complete as to v, then

$$\int h^2 = \sum_{i=1}^{\infty}\int h u_i \int v_i h \qquad (h). \tag{14}$$

These relations do not necessarily hold for every function f of $\mathfrak{F}$ unless $\mathfrak{G} = \mathfrak{F}$ and $\mathfrak{H} = \mathfrak{F}$, respectively. This is illustrated by the example at the end of this section.

Corollary 2. If the function p is orthogonal to all the functions v_i, u_i respectively, then it is orthogonal to all the functions of the classes $\mathfrak{G}$, $\mathfrak{H}$ respectively.

Corollary 3. If the biorthogonal system (v_i, u_i) is complete as to u, v, respectively, and the function p is orthogonal to all the functions v_i, u_i, respectively, then the matrices $(b)_u$ and $(b)_v$ cannot both be limited.

Theorem 9. *When the biorthogonal system (u_i, v_i) is complete as to u, v respectively there exists no function in $\mathfrak{G}$, $\mathfrak{H}$, respectively, essentially different from zero which is orthogonal to the system $\{v_i\}$, $\{u_i\}$, respectively.*

This is an immediate consequence of the equations (13) and (14).

Corollary. When $\mathfrak{G} = \mathfrak{F}$, $\mathfrak{H} = \mathfrak{F}$, respectively, a complete system $\{u_i\}$, $\{v_i\}$ respectively, implies a complete system $\{v_i\}$, $\{u_i\}$ respectively.

Theorem 10. *If the biorthogonal system (u_i, v_i) is complete as to u, v, respectively and the functions u_i, v_i, respectively, belong to the class $\mathfrak{G}$, $\mathfrak{H}$, respectively, then the system $\{v_i\}$, $\{u_i\}$ respectively is also complete.*

A function f of $\mathfrak{F}$, orthogonal to the functions v_i, would be orthogonal to the functions $\{u_i\}$ also, as is seen from the equations (11).

The following example is that of a biorthogonal system (u_i, v_i) complete as to u but not complete as to v.

Example. Let $\{\phi_i\}$ form a complete system of normalized orthogonal functions, and let $p \neq 0$ be a function belonging to $\mathfrak{F}$ but linearly dependent on no finite number of functions ϕ_i and therefore

$$\int p^2 - \sum_{k=1}^{n} \left(\int p\phi_k \right)^2 > 0 \qquad (n).$$

Construct the functions

(15)

$$u_i = \frac{\left[\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2 \right] \phi_i + \int p\phi_i \sum_{k=1}^{i-1} \phi_k \int p\phi_k}{\sqrt{\int p^2 - \sum_{k=1}^{i} \left(\int p\phi_k \right)^2} \sqrt{\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2}} \qquad (i)$$

$$v_i = u_i - \frac{p \int pu_i}{\int p^2} \qquad (i).$$

The functions u_i and v_i form a biorthogonal system (u_i, v_i) complete as to u but the function p is orthogonal to all the functions v_i. The functions v_i form a system of orthogonal and normalized functions and therefore the matrix $(b)_v$ is limited. The matrix $(b)_u$ is not limited; otherwise $\int p^2$ could be expressed by

$$\int p^2 = \sum_{i=1}^{\infty} \int pu_i \int v_i p,$$

which is impossible.

This construction could be generalized, so that the system $\{u_i\}$ would be complete and all the functions v_i orthogonal to a finite number of functions $p_1, p_2, \cdots, p_n$.

§ 6. *Associated functions.*

Definition. The function f_1 is a *vu-associated* function of the function f with respect to the biorthogonal system (u_i, v_i) if it satisfies the following relation:

$$\int f v_i = \int u_i f_1 \qquad (i). \tag{16}$$

Remark 1. If the system $\{u_i\}$ is complete, then f_1 is the only *vu*-associated function of f, and for a function p orthogonal to all the functions v_i the *vu*-associated function is zero, and the function $f + \mu p$, where μ is an arbitrary constant, has the same *vu*-associated function as f.

Remark 2. The functions v_i are *vu*-associated functions of u_i. For an orthogonal system a *vu*-associated function is the function itself.

Remark 3. The sum of two functions, which are *vu*-associated functions of two given functions, is a *vu*-associated function of the sum of the two given functions.

Definition. The function f_1 is a *uv-associated* function of the function f with respect to the biorthogonal system (u_i, v_i) if it satisfies the following relation:

$$\int f_1 v_i = \int u_i f \qquad (i). \tag{17}$$

Remark 4. A statement of the immediate consequences of this definition is obtained by interchanging u and v in the remarks (1), (2), and (3).

Theorem 11. *If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, $(b)_u$, respectively, then for every function f of $\mathfrak{F}$ there exists a vu-associated g which belongs to $\mathfrak{G}$, a uv-associated function h which belongs to $\mathfrak{H}$, respectively.*

This theorem is a consequence of the theorems 3, 5, and 6. Take any function f of $\mathfrak{F}$ and let $c_i = \int f v_i$; by theorem 3, the sequence $\{c_i\}$ is of finite norm, and by theorems 6 and 5 there exists a function g of $\mathfrak{G}$ such that

$$c_i = \int f v_i = \int u_i g \qquad (i).$$

Corollary. If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, then for every function f of $\mathfrak{F}$ there exists a *uv*-associated function when and only when the matrix $(b)_u$ is also limited.

Theorem 12. *If for a biorthogonal system, complete as to u, the vu-associated function g exists for every function f of $\mathfrak{F}$, then the matrix $(b)_v$ is limited.*

This theorem is a consequence of lemma 3 and the equations (5) and (7). Multiply the equations (5) by g and integrate and then substitute from the equations (7) for $\int gu_i$. By lemma 3 the matrix $(\sum u_{ik}u_{jk})$ is limited and by reason of the properties (2), § 2, it follows that $(a)_u$ is also limited. Corollary to theorem 4 shows that $(b)_v$ is limited.

Theorem 13. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, and if g is the vu-associated function of f, then the integral $\int fg$ can be expressed as the sum of squares:*

$$\int fg = \sum_{i=1}^{\infty} \left(\int fv_i \right)^2 \qquad (f). \tag{18}$$

This follows from a direct application of the theorem 8 and equations (11).

Remark. If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$ and if g is the vu-associated function of some function f, and the function p is orthogonal to all the functions v_i, then p is also orthogonal to g. This is a special case of the corollary to theorem 8.

Theorem 14. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, and if g_1 is the vu-associated function of f_1, and g_2 that of f_2, then*

$$\int f_1 g_2 = \int f_2 g_1 \qquad (f_1, f_2). \tag{19}$$

This is a consequence of theorem 8.

If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, then by theorem 11 the vu-associated functions define a functional transformation T,

$$g = T(f) \qquad (f), \tag{20}$$

which is applicable to the class $\mathfrak{F}$, and is single-valued if the system $\{u_i\}$ is complete.

A particular case of equation (20) is

$$v_i = T(u_i) \qquad (i). \tag{21}$$

When the system $\{u_i\}$ is complete, T is linear:

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (f_1, f_2). \tag{22}$$

In terms of T the theorems 14 and 13 may be expressed as follows:

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2) \tag{23}$$

and

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

We give below some examples of this functional transformation T.

Example 1. For a complete orthogonal system $T(f)=f$.

Example 2. For the complete biorthogonal system

$$v_i = \alpha u_i \qquad (i),$$

where α is a positive function of $\mathfrak{F}$, the transformation T is

$$T(f) = \alpha f.$$

Example 3. Let (u_i, v_i) be the biorthogonal system defined in the example in § 5. The transformation T is given by

$$T(f) = f - \frac{p\int pf}{\int p^2}.$$

Example 4. Let $K(s, t)$ be a kernel which is continuous, symmetric, has positive characteristic numbers $\{\mu_i^2\}$, and for which there exists no function f in $\mathfrak{F}$ such that $\int K(s, t)f(t)dt = 0$. Let the characteristic functions be $\{\phi_i\}$:

$$\phi_i(s) = \mu_i^2 \int K(s, t)\phi_i(t)dt \qquad (i).$$

The biorthogonal system

$$u_i = \mu_i \phi_i, \qquad v_i = \frac{\phi_i}{\mu_i} \qquad (i)$$

is complete as to u, and has a limited matrix $(b)_v$ because the sequence of numbers $\{1/\mu_i\}$ is of finite norm. The functional transformation T is

$$g(s) = \int K(s, t)f(t)dt.$$

§ 7. *Equivalent biorthogonal systems.*

Definition. The biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ complete as to u and $\bar{u}$, respectively, and satisfying the relation

$$\int u_i \bar{v}_k = \int v_i \bar{u}_k \qquad (i, k),$$

are called equivalent.

As a trivial case, all orthogonal systems are equivalent. The definition may be applied to some biorthogonal systems which are not complete.

Theorem 15. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, then two biorthogonal systems $(\bar{u}_i, \bar{v}_i)$ and $(\overline{u}_i, \overline{v}_i)$ each equivalent to (u_i, v_i) are equivalent to each other.*

This follows directly from the equations (19).

Theorem 16. *If the biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and if the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the necessary and sufficient condition that the matrix* $\int \bar{u}_i v_k$ *be orthogonal, is that the two given systems be equivalent.*

If the two systems are equivalent, the conditions for the orthogonality (§ 2) of the matrix $\int \bar{u}_i v_k$ are satisfied; for

$$\sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{v}_j = \int \bar{u}_i \bar{v}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (i,j),$$

$$\sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{u}_k v_j = \sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{v}_k u_j = \int u_i v_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (i,j).$$

To prove the converse, let $\bar{v}_i$ be the vu-associated function of $\bar{u}_i$. Then

$$\int \bar{u}_i \bar{v}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{v}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and therefore

$$\int \bar{u}_i \bar{v}_j = \int \bar{v}_j \bar{u}_i \quad (i,j).$$

Since the system $\{\bar{u}_i\}$ is complete we obtain

$$\bar{\bar{v}}_i = \bar{v}_i \quad (i)$$

and the two systems are equivalent.

Theorem 17. *If the biorthogonal system* (u_i, v_i) *is complete as to* u *and the matrix* $(b)_v$ *is limited, then for any biorthogonal system* $(\bar{u}_i, \bar{v}_i)$ *complete as to* u *and equivalent to the system* (u_i, v_i) *the matrix* $(b)_{\bar{v}}$ *is limited.*

Let f be any function of $\mathfrak{F}$ and write the equations

$$\int f \bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i \quad (i).$$

From this system of equations we see that $\left\{\int f \bar{v}_i\right\}$ is of finite norm for every f and therefore the matrix $(b)_{\bar{v}}$ is limited.

Theorem 18. *If the equivalent biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the* vu*-associated function* g *of any given function* f *is equal to the* $\bar{v}\bar{u}$*-associated function* $\bar{g}$ *of* f.

We have

$$\int \bar{g} \bar{u}_i = \int f \bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i = \sum_{k=1}^{\infty} \int g u_k \int v_k \bar{u}_i = \int g \bar{u}_i.$$

Since the system $\bar{u}_i$ is complete, $\bar{g} = g$.

Corollary. The functional transformations T and $\bar{T}$, corresponding to two biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ which satisfy the conditions of theorem 17, are equal.

Remark. If two biorthogonal systems have the same functional transformation T, they are equivalent.

§ 8. *Functional transformations T.*

In this section we consider functional transformations T, which transform every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which are single-valued and possess the following three properties: *

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (a_1, a_2, f_1, f_2), \tag{22}$$

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2), \tag{23}$$

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

Theorem 19. *Any biorthogonal system* (u_i, v_i) *such that*

$$v_i = T(u_i) \qquad (i),$$

where T *has the three properties* (22), (23), (24), *has a limited matrix* $(b)_\omega$.

Let f be any function of $\mathfrak{F}$; then by the property (24)

$$\int \left(f - \sum_{k=1}^{i} u_k \int u_k T(f) \right) \left(T(f) - \sum_{k=1}^{i} v_k \int v_k f \right) \geqq 0 \qquad (i, f).$$

Transforming this inequality, we obtain

$$\sum_{k=1}^{i} \left(\int v_k f \right)^2 \leqq \int f T(f) \qquad (i, f), \tag{25}$$

and therefore the matrix $(b)_\omega$ is limited.

Corollary. For any two functions f_1 and f_2 of $\mathfrak{F}$,

$$\left(\int f_1 T(f_2) \right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2) \qquad (f_1, f_2). \tag{26}$$

In the inequality (25) make the substitution $i = 1$, $f = f_2$, and

$$v_1 = \frac{T(f_1)}{\int f_1 T(f_1)},$$

where f_1 and f_2 are any two functions of $\mathfrak{F}$, such that $T(f_1) \neq 0$ and $T(f_2) \neq 0$.

* Since T is single-valued, the linearity (22) is a consequence of (23).

Theorem 20. *For any functional transformation T which has the three properties* (22), (23), (24), *and which does not transform every function into zero, there exist biorthogonal systems* (u_i, v_i) *for which*

$$v_i = T(u_i) \qquad (i).$$

Let $\{f_i\}$ be any sequence of linearly independent functions of $\mathfrak{F}$, for which $T(f_i) \neq 0$ and also are linearly independent. Construct the following system:

$$u_i = \frac{f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i),$$

$$v_i = \frac{T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i}{\sqrt{\int \left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i). \tag{27}$$

For $i = 1$ it is clear that the denominators do not vanish and that $v_i = T(u_i)$. Assume that the statement is true for $i - 1$; then

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = T\left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right),$$

and therefore by (24) the denominators of u and v cannot vanish unless

$$f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i) = 0$$

or

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = 0.$$

Either one of these equations is contrary to the hypothesis on $\{f_i\}$. Therefore

$$v_i = T(u_i) \qquad (i).$$

It can now be argued that the systems $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system. It is easily seen that the relations

$$\int u_i v_j = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j), \end{cases}$$

hold for $i, j = 1, 2$. Assume that they hold for $i, j = 1, 2, \cdots, n-1$ and prove that they then hold for $i, j = 1, 2, \cdots, n$.

Since in a biorthogonal system (u_i, v_i) the functions u_i and also v_i are necessarily linearly independent, the vanishing of the denominator of u_i and v_i gives a sufficient condition that $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent.

The biorthogonal system (u_i, v_i) defined by the equations (27) has by theorem 18 a limited matrix $(b)_\rho$; a vu-associated function g of any function f exists and

$$g = T(f).$$

Biorthogonal systems constructed from different sequences $\{f_i\}$ are equivalent or belong to the same type.

There exist functional transformations T with the properties (22), (23), (24) for which there does not exist a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent. For example, the functional transformation

$$T(f) = \int K(s, t) f(t)\, dt,$$

where

$$K(s, t) = \sum_{i=1}^{n} \frac{\phi_i(s)\phi_i(t)}{\mu_i^2}$$

transforms every function f into a linear combination of the n functions ϕ_i $(i = 1, \cdots, n)$. If, however, there exists a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent, then a biorthogonal system can be constructed by (27) which is complete as to u. In all cases however,

$$\int gf = \sum_{i=1}^{\infty} \int g u_i \int v_i f.$$

Theorem 21. *A single-valued functional transformation T which has the three properties* (22), (23), *and* (24), *and for which there exists a complete system of linearly independent functions $\{f_i\}$ such that $T(f_i)$ are also linearly independent, is equivalent to a type of biorthogonal systems (u_i, v_i) complete as to u and with limited matrices $(b)_\rho$.*

The formulæ for u_i and v_i may be expressed in the following form:

$$u_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_{i-1}) & f_1 \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_{i-1}) & f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_{i-1}) & f_i \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_{i-1}) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_{i-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_{i-1} T(f_1) & \int f_{i-1} T(f_2) & \cdots & \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_i) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_i) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_i) \end{vmatrix}}} \quad (i),$$

$$v_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) & T(f_i) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) & T(f_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_{i-1} T(f_i) & T(f_i) \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_1 T(f_{i-1}) & \int f_2 T(f_{i-1}) & \cdots & \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_i T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_i T(f_2) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_i T(f_i) \end{vmatrix}}} \quad (i).$$

The necessary and sufficient condition that both the systems $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent is the vanishing of the determinant

$$(28) \qquad \begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_i) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_i) \\ \cdots & \cdots & \cdots & \cdots \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_i) \end{vmatrix}$$

Theorem 22. *If, for a functional transformation T with the properties* (22), (23), *and* (24), *there exists a function p such that $p = \lambda T(p)$, then there exists a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ for which the $\bar{v}\bar{u}$-associated function g of a function f orthogonal to p is given by*

$$g = T(f).$$

Taking into consideration the inequality (26) and the condition imposed on p, we see that the functional transformation

$$\bar{T}(f) = T(f) - \frac{p \int p T(f)}{\int p^2}$$

satisfies the three conditions (22), (23), and (24). By theorem 20 we can construct a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ such that

$$\bar{v}_i = \bar{T}(\bar{u}_i).$$

The $\bar{v}\bar{u}$-associated function g of f is given by

$$g = T(f) - \frac{T(p) \int pf}{\int p^2} = T(f) - \frac{p \int p T(f)}{\int p^2}$$

and therefore if f_1 is orthogonal to p,

$$g_1 = T(f_1).$$

§ 9. *Solutions of* $f = \lambda T(f)$.

Let T be a single-valued functional transformation which transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which has the properties (22), (23), and (24); and let (u_i, v_i) be a biorthogonal system (theorem 20) for which

$$v_i = T(u_i) \qquad (i).$$

Then the matrix $(b)_a$ is limited (theorem 19) and therefore also the matrix $(a)_a$.

The matrix (α_{ik}), where

$$\alpha_{ik} = \sum_{j=1}^{\infty} a_{ij} a_{kj},$$

is limited, by lemma 6, and is symmetric.

In this section we consider the functional equation

$$f = \lambda T(f), \tag{31}$$

where λ is a parameter, and f a function to be determined.

Theorem 23. *If there exists a solution f of the equation*

$$f = \lambda T(f),$$

then

$$z_i = \int f\varphi_i \qquad (i)$$

is a solution of finite norm of the infinite system of homogeneous linear equations

$$z_i - \lambda \sum_{k=1}^{\infty} \alpha_{ik} z_k = 0 \qquad (i), \tag{32}$$

and conversely a solution of finite norm of the equations (32) *leads to a solution of the equation* (31).

The functional equation (31) is equivalent to the set of equations

$$\int f u_i = \lambda \int v_i f \qquad (i). \tag{33}$$

By means of the equations (5) and (7) we obtain the result that the equations (33) are equivalent to (32).

Corollary. The functional equation (31) has a solution f when and only when the quadratic form $\sum \alpha_{ik} x_i x_k$ has a characteristic number; the number of the solutions f depends on the multiplicity of the characteristic number.

This corollary follows from lemma 8.

Theorem 24. *Two solutions of*

$$f = \lambda T(f),$$

corresponding to two different values of λ, *are orthogonal.*

Let the two values of λ be λ_1 and λ_2, and the corresponding solutions f_1 and f_2:

$$f_1 = \lambda_1 T(f_1), \qquad f_2 = \lambda_2 T(f_2).$$

Multiply the first equation by f_2 and integrate. By means of the second equation and (23) we obtain

$$\int f_1 f_2 = \frac{\lambda_1}{\lambda_2} \int f_1 f_2,$$

and therefore

$$\int f_1 f_2 = 0,$$

when λ_1 and λ_2 are distinct.

If the kernel $K(s, t)$ is symmetric, positive semi-definite, and satisfies certain continuity conditions, the integral equation

$$f(s) = \lambda \int K(s, t) f(t)\, dt$$

is a special case of the functional equation (31). This integral equation has at least one solution, the multiplicity of a characteristic number is always finite, and if there are an infinite number of characteristic numbers the limit point is infinite.

We give examples in which the spectrum * of the quadratic form $\sum \alpha_{ik} x_i x_k$ consists of a point spectrum with finite or infinite multiplicity, or with a finite or infinite limit point, or of a continuous spectrum.

Example 1. Same as the example in § 5.

The spectrum consists of the one point unity with an infinite multiplicity. Any function orthogonal to p is a solution of the equation (31).

Example 2. Let $\{\phi_i\}$ be a complete system of normalized and orthogonal functions; and let (u_i, v_i) be the biorthogonal system defined by

$$u_i = \sum_{k=1}^{i} \phi_k \qquad (i),$$

$$v_i = \phi_i - \phi_{i+1} \qquad (i).$$

The quadratic form $\sum \alpha_{ik} x_i x_k$ becomes

$$x_1^2 + 2 \sum_{i=2}^{\infty} x_i^2 - 2 \sum_{i=1}^{\infty} x_i x_{i+1}$$

* *H.* IV, p. 172.

and its spectrum* is a continuous spectrum consisting of the values λ between $+1$ and $+\infty$. The equation (31) has no solution f belonging to $\mathfrak{F}$.

Example 3. Same as example 1, § 4, in which the constants a_i approach zero as a limit, and the functions ϕ_i are complete and continuous.

The spectrum consists of a point spectrum with limit point at 1:

$$\lambda_{2j-1} = \frac{2 + a_j^2 + a_j\sqrt{a_j^2+4}}{2}, \qquad \lambda_{2j} = \frac{2 + a_j^2 - a_j\sqrt{a_j^2+4}}{2} \qquad (j).$$

The corresponding solutions of (31) are

$$\alpha_{2j-1} = \frac{\phi_{2j-1}(1-\lambda_{2j-1}) - a_j\phi_{2j}}{(1-\lambda_{2j-1})^2 + a_j^2}, \qquad \alpha_{2j} = \frac{\phi_{2j-1}(1-\lambda_{2j}) - a_j\phi_{2j}}{(1-\lambda_{2j})^2 + a_j^2} \qquad (j).$$

The functions $\{\alpha_i\}$ form a complete system of normalized orthogonal functions.

It is possible to take a_i so small that

$$K(s,t) = \sum_{i=1}^{\infty} \frac{\alpha_i(s)\alpha_i(t)}{1-\lambda_i}$$

converges uniformly; $1/1-\lambda_i$ are then the characteristic numbers of $K(s,t)$. With this restriction imposed on a_i, we obtain for any function f and its vu-associated function g

$$f(s) = g(s) - \int K(s,t)g(t)dt.$$

Since 1 is not a characteristic number of $K(s,t)$ we can solve this equation for g and obtain for T

$$g(s) = f(s) - \int k(s,t)f(t)dt,$$

where $k(s,t)$ is the resolvent of $K(s,t)$. The first equation gives the relation between any function g and its uv-associated function f.

Example 4. Same as example 1, § 4, in which however the constants a_i are all equal to the constant a, the system $\{\phi_i\}$ is complete, and the functions ϕ_i are continuous.

In this case the spectrum consists of the two points:

$$l = \frac{2 + a^2 + a\sqrt{a^2+4}}{2}, \qquad \frac{1}{l} = \frac{2 + a^2 - a\sqrt{a^2+4}}{2},$$

* H. WEYL, *Singuläre Integralgleichungen*, Dissertation, Göttingen, 1908, p. 69.

each of infinite multiplicity. Let the solutions of (31), corresponding to l and $1/l$, be $\{\alpha_i\}$ and $\{\alpha_i\}$ respectively. Each of these systems we may consider as orthogonal and normalized; and each system is orthogonal to the other. Further, the two systems together form a complete orthogonal system of continuous functions; and it is possible to choose systems of constants $\{l_i\}$ and $\{l_i^*\}$ such that they are the characteristic numbers of two symmetric kernels $K(s, t)$ and $K^*(s, t)$, whose corresponding characteristic functions are the $\{\alpha_i\}$ and $\{\alpha_i^*\}$ respectively. Then the functions u_i and v_i satisfy the equation

$$\int [K(s,t) + lK^*(s,t)]\, u_i(t)\,dt = \int [lK(s,t) + K^*(s,t)]\, v_i(t)\,dt \quad (i).$$

This is an integral equation of the first kind in both u_i and v_i. The functional transformation is given by*

$$v_i(s) = 2\int_0^\infty \left(\sum_k (-1)^k \frac{t^{2k+1}}{k!}\, \beta_{2k+1}^{(i)}(s)\right) dt \qquad (i),$$

where

$$\beta_k^{(i)}(s) = l\int [l^{2k} K^{(2k+2)}(s,t) + K^{*(2k+2)}(s,t)]\, u_i(t)\,dt$$

and $K^{(2k+2)}(s, t)$ and $K^{*(2k+2)}(s, t)$ are iterated kernels of K and K^* respectively.

§ 10. *Complete† elementary theory of six properties.*

A biorthogonal system (u_i, v_i) may have one or more of the following six properties: 1) $\{u_i\}$ complete; 2) $\{v_i\}$ complete; 3) $(a)_u$ limited; 4) $(b)_u$ limited; 5) $(a)_v$ limited; 6) $(b)_v$ limited. If we consider all the possibilities of a biorthogonal system possessing or not possessing each one of these six properties. there are $2^6 = 64$ cases to be considered. To establish a complete elementary theory we must, for each of the 64 cases, either show by theorems the impossibility of the case or exhibit an example. In this way we show that there are no general theorems relating to these six properties other than those already obtained.

Theorem 25. *There are no other theorems which are analogous to the corollary to theorem 4, theorem 6, and corollary 3 to theorem 8, and which express interdependence among the six properties mentioned above.*

To denote that a biorthogonal system possesses or does not possess the ith property $(i = 1, 2, \cdots, 6)$ we write $+$ or $-$ in the ith place.

Since by theorem 6 a limited matrix $(b)_v$, $(b)_u$ respectively implies a limited matrix $(a)_u$, $(a)_v$ respectively, the following cases are excluded

$$(\pm, \pm, -, \pm, \pm, +),$$
$$(\pm, \pm, \pm, +, -, \pm),$$

of which 28 are distinct.

* H. BATEMAN, *Inversion of a definite integral*, Mathematische Annalen, vol. 63 (1907).

† E. H. MOORE, *Introduction to a form of General Analysis*, The New Haven Mathematical Colloquium, p. 82.

The corollary to theorem 4 excludes the cases

$$(\pm, \pm, +, \pm, \pm, -),$$

$$(\pm, \pm, \pm, -, +, \pm).$$

Of these there are 11 cases which are new and distinct.

Corollary 3 to theorem 8 excludes the two cases

$$(+, -, +, +, +, +),$$

$$(-, +, +, +, +, +).$$

There remain 23 cases, for which we exhibit examples. Two cases which differ only by an interchange of u and v may be considered as one case.

1) $(+, +, +, +, +, +)$.

Example 1 in § 4, in which the system $\{\phi_i\}$ is complete.

This example shows the impossibility of a theorem to the effect that a biorthogonal system complete both as to u and v cannot have all the matrices limited.

2) $(+, +, +, -, -, +)$,

3) $(+, +, -, +, +, -)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

4) $(+, +, -, -, -, -)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

5) $(-, -, +, +, +, +)$.

Example 1 in § 4, in which the system $\{\phi_i\}$ is incomplete.

6) $(-, -, -, +, +, -)$,

7) $(-, -, +, -, -, +)$.

Example 3 in § 4, in which the system $\{\phi_i\}$ is incomplete.

8) $(-, -, -, -, -, -)$.

Example 2 in § 4, in which the system $\{\phi_i\}$ is incomplete.

9) $(-, -, +, +, +, -)$,

10) $(-, -, +, -, +, +)$.

Example 4 in § 4, in which the system $\{\phi_i\}$ is incomplete.

11) $(-, -, +, -, -, -)$,

12) $(-, -, -, -, +, -)$.

Examples for the cases 20) and 21) below, in which the functions u_1 and v_1 are omitted.

13) $\quad (-, -, +, -, +, -).$

$\{\phi_i\}$ is an incomplete system of normalized orthogonal functions. $\{a_i\}$ is a sequence of constants such that $\{1/a_i\}$ is of finite norm. $\{\lambda_i\}$ is a sequence of constants such that $\{1/\lambda_i\}$ is of finite norm and also $\{\lambda_i/a_i\}$ is of finite norm. The biorthogonal system

$$u_i = \lambda_i \phi_{2i-1}, \qquad v_i = (\phi_{2i-1} + a_i \phi_{2i}) \frac{1}{\lambda_i} \qquad (i)$$

clearly has a limited matrix $(a)_u$ and unlimited matrices $(b)_u$ and $(b)_v$. That the matrix $(a)_v$ is limited follows from the fact that, for the biorthogonal system $(\bar{u}_i, \bar{v}_i)$:

$$\bar{u}_i = \frac{\lambda_i \phi_{2i}}{a_i}, \qquad \bar{v}_i = v_i \qquad (i),$$

the matrix $(b)_{\bar{u}}$ is limited.

14) $\quad (+, -, +, -, +, +),$

15) $\quad (-, +, +, +, +, -).$

Example in § 5,

16) $\quad (+, -, +, -, -, +),$

17) $\quad (-, +, -, +, +, -).$

The biorthogonal system is given by

$$u_i = \lambda_i \bar{u}_i, \qquad v_i = \frac{\bar{v}_i}{\lambda_i} \qquad (i),$$

where $(\bar{u}_i, v_i)$ is the biorthogonal system in the example in § 5, and the sequence $\{1/\lambda_i\}$ is of finite norm.

18) $\quad (+, -, -, +, +, -),$

19) $\quad (-, +, +, -, -, +).$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \qquad (i),$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5, and λ_i are chosen so that $(b)_v$ is not limited and $(b)_u$ is limited.

20) $\quad (+, -, -, -, +, -),$

21) $\quad (-, +, +, -, -, -).$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i - p}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \qquad (i),$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5 and the λ_i are constants such that $\{1/\lambda_i\}$ is not, but $\{1/\lambda_i^2\}$ is, of finite norm.

22) $\qquad (+, -, -, -, -, -),$

23) $\qquad (-, +, -, -, -, -).$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{u}_i \sum_{k=1}^{i-1}\bar{u}_k\int p\bar{v}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

$$v_i = \frac{\left(\bar{v}_i - \frac{p\int p\bar{v}_i}{\int p^2}\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{v}_i \sum_{k=1}^{i-1}\left(\bar{v}_k - \frac{p\int p\bar{v}_k}{\int p^2}\right)\int p\bar{u}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

where

$$\begin{aligned} \bar{u}_{2j} &= a_j\phi_{2j}, & \bar{u}_{2j+1} &= \phi_{2j+1} + a_j^2\phi_{2j}, \\ \bar{v}_{2j} &= \frac{\phi_{2j} - a_j^2\phi_{2j+1}}{a_j}, & \bar{v}_{2j+1} &= \phi_{2j+1} \end{aligned} \qquad (j),$$

and $\{\phi_i\}$ is a complete system of normalized orthogonal functions, $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm, and p a function such that

$$\int p\phi_{2j} = \frac{1}{a_j^3}, \qquad \int p\phi_{2j+1} = \frac{1}{a_j^5}.$$

The two matrices $(b)_{\bar{u}}$ and $(b)_{\bar{v}}$ are unlimited. The function p is orthogonal to all the functions v_i; the system $\{u_i\}$ is complete. The matrices $(b)_v$ and $(a)_u$ are unlimited for $\int p\bar{v}_{2j} = 0$ and therefore $v_{2j} = \bar{v}_{2j}$. Functions f orthogonal to p can be found for which $\left\{\int fu_i\right\}$ is not of finite norm and therefore $(b)_u$ is unlimited. By considering the biorthogonal system $(\bar{u}_i, \bar{v}_i)$:

$$\begin{aligned} \bar{u}_1 &= \frac{p}{\sqrt{\int p^2}}, & \bar{u}_{i+1} &= u_i - \frac{p\int pu_i}{\int p^2}, \\ \bar{v}_1 &= \frac{p}{\sqrt{\int p^2}}, & \bar{v}_{i+1} &= v_i \end{aligned} \qquad (i),$$

which is complete both as to $\bar{u}$ and as to $\bar{v}$, we find that $(a)_v$ is unlimited.

March, 1910.

APPLICATIONS OF BIORTHOGONAL SYSTEMS OF FUNCTIONS TO THE THEORY OF INTEGRAL EQUATIONS*

BY

ANNA JOHNSON PELL

Introduction.

In this paper we give a sufficient condition that the characteristic numbers of an unsymmetric kernel exist and be real, and prove the expansibility of arbitrary functions in terms of the corresponding characteristic functions. This sufficient condition is stated in terms of a functional transformation† $T(f)$ defined by certain general properties (§ 1), and for the special case $T(f)=f$ we obtain the known theory of the orthogonal integral equations. The method employed is that of infinitely many variables and is based to some extent on an earlier paper.‡ However, the present paper, with the exception of a few footnotes, can be read independently of (I) if Theorems 20 and 8 which are here stated in § 1 are accepted; and if the suggestion in Remark 1 of § 1 is carried out, the only reference to (I) that is necessary is Theorem 20.

§ 1. *Preliminary notation and theorems.*

We denote by I the interval $a \leqq s \leqq b$ of the real variable s, and by S the square

$$a \leqq s \leqq b, \qquad a \leqq t \leqq b,$$

* Presented to the Society, September, 1909.

† Since the original manuscript (the present paper is essentially unchanged in content and method, but somewhat revised in form) was sent to the editors in March, 1910, J. MARTY has announced some results for unsymmetric kernels in the Comptes Rendus. His sufficient condition (April, 1910), which is also necessary (June, 1910) is stated in terms of a special $T(f)$, namely $T(f)=\int K(s,t)f(t)dt$, which satisfies the conditions of § 1 imposed on the $T(f)$ of this paper. Special examples show that the condition (c) of § 1 is more or less essential, see also MARTY's correction (ibid., June, 1910). The method indicated by him is a generalization of that used by E. SCHMIDT to prove the existence and is not essentially different from the method referred to by the author in the footnote at the beginning of § 3. MARTY states no expansion theorems. In May, 1910, the author sent a note to the Bulletin of the American Mathematical Society which appeared in July, 1910, and which gives necessary and sufficient conditions in terms of the general $T(f)$.

‡ A. J. PELL, these Transactions, this number. It will be referred to as (I).

of the real variables s and t. We denote by $\mathfrak{F}$ the class of all real functions of s which are integrable and whose squares are integrable on the interval I in the sense of LEBESGUE, and regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero. The argument of the function is omitted whenever possible. The integral

$$\int_a^b \varphi(s)\,ds$$

is denoted by $\int \varphi$. A sequence of real numbers is of finite norm if the sum of the squares is convergent.

Let $T(f)$ be a single valued functional transformation with the following properties:

(*a*) $T(f)$ transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$.

(*b*) For any two functions f_1 and f_2 of $\mathfrak{F}$,

$$\int f_1\, T(f_2) = \int f_2\, T(f_1).$$

From this property follows the linearity of $T(f)$:

$$T(a_1 f_1 + a_2 f_2) = a_1\, T(f_1) + a_2\, T(f_2).$$

(*c*) For every function f of $\mathfrak{F}$,

$$\int f T(f) \geqq 0,$$

the equality sign holding only for functions f such that $T(f) \equiv 0$.

Theorems 20 and 8 of (I) give the existence* of a biorthogonal system (U_i, V_i) such that

(*d*) $$V_i = T(U_i).$$

(*e*) For every function f of $\mathfrak{F}$,

$$\sum_i \left(\int f V_i\right)^2 \leqq \int f T(f).$$

(*f*) For any functions† f_1 and $g_2 = T(f_2)$,

$$\int f_1 g_2 = \sum_i \int f_1 V_i \int U_i g_2.$$

For the purposes of this paper we suppose further that $T(f)$ has the following properties:

* The theory developed in (I) shows that every biorthogonal system (u_i, v_i) such that $\sum_i \left(\int f v_i\right)^2$ converges for every f of $\mathfrak{F}$ gives rise to a functional transformation $T(f)$ with the properties (*a*), (*b*), (*c*) and such that $v_i = T(u_i)$.

† Theorem 8 of (I) is more general than this in some cases.

(a_1) $T(f)$ transforms every continuous function into a continuous function; and for every continuous function f

$$\int |T(f)| \leqq k \int |f|,$$

where k is a constant independent of f.

(c_1) There is at most* one function p, not essentially zero, such that $T(p) \equiv 0$ and this function p is continuous.

In virtue of (a_1) and (c_1) we can assume that the biorthogonal system (U_i, V_i) is such that

(h) The functions U_i and V_i are continuous and the system U_i is complete.

Remark 1. If the conditions on $T(f)$ were stated for continuous functions only in the form of (a_1), (b) and (c), we could exhibit a biorthogonal system (u_i, v_i) such that the properties (d), (e) and (h) hold for continuous functions by referring to Theorem 20 of (I), and the property (f) could be shown to hold for all continuous functions f_1 and f_2 by a simple generalization of HILBERT'S proof for the analogous theorem for orthogonal functions (H, V, pp. 443–445).† The generalization consists of the systematic substitution of $\int fT(f)$ for $\int f^2$.

Remark 2. If $K(s, t)$ is continuous on S, then the transformed function with respect to either variable is also continuous on S. This follows from the conditions ‡ (a_1) and (b) and a theorem § by Professor E. H. MOORE, namely; the necessary and sufficient condition that the sequence of continuous functions $f_n(s)$ converge uniformly is, that for every e there exist an n_e depending on e such that for $n_1 \geqq n_e$ and $n_2 \geqq n_e$

$$\left| \int (f_{n_1} - f_{n_2}) f \right| \leqq k_1 \int |f|$$

for every continuous function f.

Remark 3. For every function $K(s, t)$ continuous on S, we have

$$T_s \int K(s, t)\,dt - \int T_s K(s, t)\,dt = 0,$$

---

*Throughout the paper p denotes a function such that $T(p) \equiv 0$. The condition (c_1) is added for simplicity; the results obtained would hold with slight modifications if there were more than one such function p. See Remark (7), § 3.

† HILBERT'S fourth and fifth memoirs in Göttinger Nachrichten are referred to as H, IV, and H, V.

‡ These conditions are used as follows:

$$\left| \int f(t)[T_t K(s', t) - T_t K(s, t)]\,dt \right| = \left| \int T[f(t)][K(s', t) - K(s, t)]\,dt \right|$$

$$\leqq k_1 \int |T(f)| \leqq k k_1 \int |f|.$$

§ *Introduction to a form of General Analysis*, in The New Haven Mathematical Colloquium (1910), p. 5; Atti del IV congresso internazionale dei matematici, vol. 2 (1909), p. 103.

where T_s denotes that T operates on the argument s. To prove this, multiply the left-hand side of the equation by an arbitrary continuous function, integrate and apply property (b).

In a similar manner it can be shown that

$$T_s T_t K(s, t) = T_t T_s K(s, t).$$

A particular case of this is that $T_s T_t K(s, t)$ is symmetric if $K(s, t)$ is symmetric.

§ 2. *Integral equation with general kernel.*

Let $K(s, t)$ be a function continuous in S, f a function continuous on I, $T(f)$ the functional transformation defined in § 1and u an unknown function. Consider the integral equations

$$u(s) + \int K(s, t)\, T[u(t)]\, dt = f(s), \tag{1}$$

$$u(s) + \int K(s, t)\, T[u(t)]\, dt = 0, \tag{2}$$

and their adjoints

$$u(s) + \int K(t, s)\, T[u(t)]\, dt = f(s), \tag{3}$$

$$u(s) + \int K(t, s)\, T[u(t)]\, dt = 0. \tag{4}$$

By means of the biorthogonal system (U_i, V_i) defined in § 1 we establish a one-to-one correspondence between the equations (1), (2), (3), (4) and a system of linear equations in infinitely many variables α_i,

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji} \alpha_j = a_i, \tag{5}$$

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji} \alpha_j = 0, \tag{6}$$

and their adjoints. In these equations the sequence $\{a_i\}$ is of finite norm and

$$k_{ij} = \int\int K(s, t) V_i(t) V_j(s)\, ds\, dt.$$

We proceed by forming the following functions and constants

$$k_i(s) = \int K(s, t) V_i(t)\, dt, \qquad k_{ij} = \int k_i V_j, \qquad a_i = \int f V_i,$$

$$L(s, t) = T_t K(s, t), \qquad L_1(s, t) = T_s K(s, t),$$

$$M(s, t) = T_s L(s, t) = T_t L_1(s, t),$$

$$l_i(s) = T[k_i(s)] = \int L(s, t) V_i(t)\, dt = \int M(s, t) U_i(t)\, dt.$$

The functions k_i and l_i are continuous on I and $L(s,t)$, $L_1(s,t)$ and $M(s,t)$ are continuous on S. From the property (f) of § 1 we obtain

$$\sum_{j=1}^{\infty}[k_j(s)]^2=\int K(s,t)L(s,t)dt,$$

$$\sum_{j=1}^{\infty}[l_j(s)]^2=\int M(s,t)L_1(s,t)dt,$$

$$\sum_{j=1}^{\infty}k_{ij}^2=\int k_i l_i,$$

and since

$$\sum_{i=1}^{n}\sum_{j=1}^{\infty}k_{ij}^2=\sum_{i=1}^{n}\int k_i l_i\leqq\sum_{i=1}^{n}\int(k_i^2+l_i^2)$$

we obtain finally

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}k_{ij}^2\leqq\int\int K(s,t)L(s,t)ds\,dt+\int\int M(s,t)L_1(s,t)ds\,dt.$$

This last inequality shows that the bilinear form

$$\sum k_{ij}x_i y_j$$

is continuous (*H*, IV, p. 203). Therefore Hilbert's theorems (*H*, IV, p. 219, and *H*, V, p. 449) can be applied to the equations (5) and (6).

Theorem 1. *The non-homogeneous equation* (1) *has a unique continuous solution if the homogeneous equation* (2) *has no solution. If the homogeneous equation* (2) *has a continuous solution* u, *it is of finite multiplicity* n, *and the adjoint equation* (4) *has the same number* n *of linearly independent solutions* $\bar{u}$; *and the necessary and sufficient conditions that the non-homogeneous equations* (1) *and* (3) *have solutions are respectively*

$$\int fT\bar{u}_k=0 \qquad (k=1,2,\cdots,n),$$

and

$$\int fT(u_k)=0 \qquad (k=1,2,\cdots,n).$$

After the correspondence has been established the proof follows from an application of Hilbert's theorems mentioned above. If the equation (5) has a solution $\{\alpha_i\}$ of finite norm the function

$$\alpha=\sum_{j=1}^{\infty}\alpha_j k_j \tag{7}$$

is continuous, since from property (e) the series converges uniformly (*H*, V, p. 442). Multiply (7) by V_i and integrate,

$$\int\alpha V_i=\sum_{i=1}^{\infty}\alpha_j k_{ji}=a_i-\alpha_i.$$

Let

$$u = f - \alpha,$$

then

$$\int u V_i = \alpha_i.$$

The function u is continuous and is a solution of the equation (1), for making the substitution

$$f_1(t) = K(s, t), \qquad g_2 = T(u)$$

in the equation (f), § 1, we obtain

$$\int K(s, t)\, T[u(t)]\, dt = \sum_{j=1}^{\infty} \alpha_j k_j(s) = f(s) - u(s).$$

Conversely if u is a continuous solution of (1), the sequence of constants

$$\alpha_i = \int u V_i$$

is of finite norm and satisfies the equation (5), for the series

$$\sum_{i=1}^{\infty} \alpha_i k_i(s) = \int K(s, t) T[u(t)]\, dt$$

is uniformly convergent, and after multiplication by V_i and integration, we obtain the equation (5).

In exactly the same manner the equivalence * of the other sets of equations can be shown.

Applying property (b) of $T(f)$ on the equations (1) and (2), and operating with T on the equations (3) and (4), theorem 1 can be argued from the Fredholm theory.

§ 3. *Integral equation with symmetric kernel.*†

From the hypothesis

$$K(s, t) = K(t, s)$$

follows immediately

$$k_{ij} = k_{ji},$$

* In virtue of theorem 16, (I), any other biorthogonal system of functions satisfying (d), (e) and (f) leads to the same solutions of the integral equations as that obtained by using the system (U_i, V_i) (H, V, p. 451).

† The results obtained in §§ 3, 4 could be obtained by a proper generalization of the method developed by E. SCHMIDT in his dissertation and in Mathematische Annalen, vol. 63. In place of SCHWARZ's inequality we have

$$\left(\int f_1 T(f_2)\right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2)$$

(see § 8, I), and for the iterated kernels

$$K^{(n)}(s, t) = \int K(s, r)\, T_r K^{n-1}(r, t)\, dr.$$

where the constants k_{ij} are defined in § 2. By means of an orthogonal transformation of the variables* $\{x_i\}$ the continuous (§ 2) quadratic form,

$$K(x) = \sum k_{ij} x_i x_j,$$

if one assumes that not all of the coefficients k_{ij} are zero, is transformed into the normal form (*H*, IV, p. 201):

$$K(x) = \sum_{i=1}^{\infty} c_i x_i'^2.$$

All of the coefficients c_i are real. But some may have the value zero, let the corresponding linear forms be

$$x'_{h_i} = \sum_{k=1}^{\infty} m_{ik} x_k = M_i(x).$$

Denote the other values of c_i by $1/\lambda_i$, and the corresponding linear transformations by

$$x'_{g_i} = \sum_{k=1}^{\infty} l_{ik} x_k = L_i(x).$$

The linear forms $L_i(x)$ and $M_i(x)$ are connected by the following relations (*H*, IV, p. 202):

$$\sum_{j=1}^{\infty} l_{ij} l_{kj} = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k), \end{cases} \tag{8}$$

$$\sum_{j=1}^{\infty} l_{ij} m_{kj} = 0, \tag{9}$$

$$\sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} L_i(x) L_i(y) + \sum_{i=1}^{\infty} M_i(x) M_i(y). \tag{10}$$

The relations connecting the linear forms $L_i(x)$ and $M_i(x)$ and the quadratic form $K(x)$ are as follows:

$$\sum_{j=1}^{\infty} l_{jk} k_{ji} = \frac{l_{ki}}{\lambda_k}, \tag{11}$$

$$\sum_{j=1}^{\infty} m_{jk} k_{ji} = 0. \tag{12}$$

Remark 4. We need to consider the exceptional cases $k_{ij} = 0$ for all i and j. Since

$$k_{ij} = \int\int K(s, t) V_i(s) V_i(t) ds\, dt = \int\int T_s T_t K(s, t) U_i(s) U_i(t) ds\, dt,$$

it follows from (h) that

$$T_s T_t K(s, t) \equiv 0,$$

* By $\{x_i\}$ and $\{y_i\}$ we denote sequences of finite norm.

and therefore, if there exists a p, $K(s, t)$ has the form

$$K(s, t) = \alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

where α is any continuous function and k is any constant.

Theorem 2. *For a symmetric continuous kernel $K(s, t)$, which is not identically zero and which is not of the form $\alpha(s)p(t)+\alpha(t)p(s)+kp(s)p(t)$ if there is a function p, there exists at least one value of λ which is necessarily real and for which the integral equation*

$$u(s) = \lambda \int K(s, t)\, T[u(t)]\, dt \tag{13}$$

has a continuous solution $u(s)$ which is not identically zero. If there is a p and if $K(s, t)$ has the form

$$\alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

there are no solutions u.

The theorem follows directly from the existence of the constants λ_i and the solutions l_{ik} of (11), and an application of the process employed in § 2. We put

$$u_i(s) = \lambda_i \sum_{j=1}^{\infty} l_{ij} k_j(s) \tag{14}$$

and show that

$$u_i(s) = \lambda_i \int K(s, t) T[u_i(t)]\, dt.$$

Remark 5. If there is a function p, then the existence theorem for the equation (13) is equivalent to an existence theorem for the equation

$$w(s) + \mu p(s) = \lambda \int K(s, t) T[w(t)]\, dt, \qquad \int wp = c, \tag{15}$$

where μ is a parameter and c is any given constant.

Corollary 1. There exists at least one characteristic number λ, which is necessarily real, for an unsymmetric kernel $L(s, t)$ which is of the form $T_t K(s, t)$, where $K(s, t)$ is symmetric and is not of the form

$$\alpha(t)p(s) + \alpha(s)p(t) + kp(s)p(t)$$

if a function p exists. Corresponding to the characteristic numbers λ_i the kernel $L(s, t)$, and the transposed kernel $L(t, s)$, have continuous characteristic functions u_i, and v_i, respectively, which form a biorthogonal system (u_i, v_i) of the type* $T(f)$.

Applying property (b) to the right-hand side of the equation (13) we obtain

$$u(s) = \lambda \int L(s, t) u(t)\, dt. \tag{16}$$

* That is, $v_i = T(u_i)$.

Operating with $T(f)$ on the equation (15) we obtain after substituting $v = T(u) = T(w + \mu p)$

$$v(s) = \lambda \int L(t, s) v(t) dt. \tag{17}$$

In both cases the processes are reversible. That the functions u_i and v_i form a biorthogonal system follows from the equations (8).

Remark 6. The existence theorem for the equation (15) is equivalent to an existence theorem for the equations

$$T(w) = \lambda \int M(s, t) w(t) dt, \qquad \int wp = c, \tag{18}$$

where c is any given constant, and $M(s, t) = T_s T_t K(s, t)$ and is therefore symmetric (Remark 3).

Remark 7. If there were more than one function p, the exceptional cases in theorem 2 would be given * by $T_s T_t K(s, t) \equiv 0$.

If the functions $f(s)$ and $K(s, t)$ were integrable and with integrable squares, the theorems 1 and 2 would hold for any functional transformation $T(f)$ with the properties (a), (b) and (c); the solutions being integrable with integrable squares.

§ 4. *Development of arbitrary functions.*

Theorem 3. *Any function f expressible in the form* †

$$f(s) = \int K(s, t) g_1(t) dt = \int L(s, t) f_1(t) dt,$$

where f_1 is any continuous function and $g_1 = T(f_1)$, can be developed into the uniformly convergent series

$$f(s) = \sum_i u_i(s) \int f v_i + v_f p(s). \tag{19}$$

In the linear form $M(x)$ make the substitution

$$x_i = k_i(s);$$

the resulting series is uniformly convergent. Multiply by V_j and integrate; from the relation (9) we obtain

$$\int M_i[k(s)] V_j(s) ds = 0$$

* The functions U_i and p_i would form a complete system.

† The theorem could be stated for any function g_1 such that $\Sigma\left(\int g_1 u_i\right)^2$ converges, or even more broadly in some cases (Theorem 8, (I)).

and therefore, if there is a p,

$$M_i[k(s)] = \sum_{j=1}^{\infty} m_{ij} k_j(s) = \nu_i p(s),$$

otherwise, $M_i[k(s)] = 0$.

In the equation (10) make the substitution

$$x_i = \int g_1 U_i, \qquad y_i = \int K(s, t) V_i(t) dt.$$

The result is

$$f(s) = \int K(s, t) g_1(t) dt = \int L(s, t) f_1(t) dt$$

$$= \sum_i \left(\sum_{j=1}^{\infty} l_{ij} \int g_1 u_j \right) \left(\sum_{j=1}^{\infty} l_{ij} k_j(s) \right) + \nu_f p(s)$$

$$= \sum_i \frac{u_i(s) \int g_1 u_i}{\lambda_i} + \nu_f p(s)$$

$$= \sum_i u_i(s) \int v_i f + \nu_f p(s).$$

Theorem 4. *Any function f expressible in the form*

$$f(s) = \int M(s, t) f_1(t) dt = \int L(t, s) g_1(t) dt,$$

where f_1 is any continuous function and $g_1 \equiv T(f_1)$, can be developed into the uniformly convergent series*

$$f(s) = \sum_i v_i(s) \int u_i f. \tag{20}$$

This development may be obtained either by substituting in the equation (10)

$$x_i = \int f_1 V_i, \qquad y_i = \int M(s, t) U_i(t) dt,$$

or by operating on the development in theorem 3 with the transformation $T(f)$.

Remark 8. The kernel $L(s, t)$ has no characteristic number λ not equal to λ_i; suppose it had, and let u be the corresponding characteristic function, then u could be developed into a uniformly convergent series, but the coefficients $\int uv_i$ would be zero, and therefore $u = 0$ or $u = p$.

* This restriction on the function g_1 is not necessary. See footnote to theorem 3.

Similarly we can show that there are no solutions corresponding to a characteristic number λ_i, which are not linear in the characteristic functions belonging to λ_i.

Remark 9. If there are only a finite number n of characteristic numbers λ_i it follows from the developments (19) and (20) that the kernels have the form

$$K(s,t)=\sum_{i=1}^{n}\frac{u_i(s)u_i(t)}{\lambda_i}+\alpha(s)p(t)+\alpha(t)p(s)+kp(s)p(t),$$

$$L(s,t)=\sum_{i=1}^{n}\frac{u_i(s)v_i(t)}{\lambda_i}+p(s)\beta(t),$$

$$M(s,t)=\sum_{i=1}^{n}\frac{v_i(s)v_i(t)}{\lambda_i}.$$

Theorem 5. *The maximum value which the expression*

$$\left|\int\int K(s,t)\,T[f(s)]\,T[f(t)]\,dsdt\right|$$

can assume for all continuous functions f such that

$$\int fT(f)=1$$

is equal to the reciprocal of that λ_i which is smallest in absolute value, and this maximum is attained for the corresponding characteristic function.

§ 5. *Examples.*

As examples of the different forms of integral equations for which § 3 gives existence theorems we give the following:

Example 1.

$$u(s)=\lambda\int K(s,t)\alpha(t)u(t)dt,$$

where α is a continuous positive function.

Example 2.

$$u(s)=\lambda\int\int K(s,t)K^*(s,t)u(t_1)dt\,dt_1,$$

where $K^*(s,t)$ is a positive-definite continuous kernel.

Example 3.

$$u(s)+\mu q(s)=\lambda\int K(s,t)T[u(t)]dt,\qquad \int uq=0,$$

where q is an an invariant function of the functional transformation $T(f)$, i. e., $q=kT(q)$. The equations are derived from the equations (15) where

$c = 0$ and the functional transformation * is

$$\bar{T}(f) = T(f) - \frac{q \int q T(f)}{\int q^2},$$

which satisfies all the conditions of § 1.

As a special case of this, for $T(f) = f$, we have the integral equation treated by CAIRNS in his dissertation. †

Example 4. Problems in the calculus of variations in which the condition

$$\int f T(f) = 1$$

replaces the usual condition

$$\int f^2 = 1,$$

give rise to differential-integral equations of the form

$$L(u) + \lambda T(u) = 0,$$

where

$$L(u) = \frac{d\left(p_1 \frac{du}{ds}\right)}{ds} + p_2 u(s),$$

and p_1 is a positive function which together with its first derivative is continuous, and p_2 is a continuous function. The equation above reduces to the integral equation

$$u(s) = \lambda \int G(s, t)\, T[u(t)]\, dt,$$

where $G(s, t)$ is a Green's function for $L(u)$ satisfying certain boundary conditions. This equation has infinitely many solutions, satisfying the same boundary conditions as $G(s, t)$, corresponding to real values of λ, if there are only a finite number of functions p_i such that $T(p_i) \equiv 0$.

A special case is the non-self-adjoint differential equation

$$L[\bar{L}(v)] - \lambda v = 0,$$

where $\bar{L}(v)$ is a self-adjoint differential expression. The adjoint differential equation is

$$\bar{L}[L(u)] + \lambda u = 0.$$

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* See Theorem 16, (I).

† W. D. CAIRNS, *Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen*, Göttingen, 1907.

Example 5. In example (2) let the function $K^*(s, t)$ be equal to the Green's function $G(s, t)$ of example (4). Then there exists at least one value of λ for which the equation

$$L(v) + \lambda \int K(s, t)v(t)dt = 0$$

has a solution v which is continuous together with its first and second derivatives and satisfies the boundary conditions of $G(s, t)$.

The two systems ψ_i and $-\psi_i''$, considered by E. SCHMIDT,* form a biorthogonal system of the type

$$T(f) = \int G(s, t)f(t)dt,$$

where $G(s, t)$ is the Green's function, for $L(v) = -v''$, which vanishes at the ends of the interval.

Example 6. Let $\bar{T}(f)$ be a functional transformation which satisfies the conditions (a), (b), and (a_1) of § 1, but not necessarily (c) and (c_1). Let $K(s, t)$ be a symmetric, continuous kernel which is positive-semi-definite:

$$\int\int_t K(s, t)f(s)f(t)ds\,dt \geqq 0,$$

the equality sign holding at most for one function p. Then the functional transformation

$$T(f) = \int K(s, t)f(t)dt$$

satisfies all the conditions of § 1.

If the integral equation †

$$(A) \qquad u(s) = \lambda \int \bar{T}_t K(s, t)u(t)dt$$

has a solution u for some value of λ, then the adjoint integral equation

$$v(s) = \lambda \int [\bar{T}_t K(s, t)]\, v(t)dt$$

has a solution v such that

$$v = T(u).$$

Two sets of solutions (u_i, v_i) and (u_j, v_j) corresponding to two different values of λ satisfy the condition

$$\int u_i v_j = 0;$$

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* Mathematische Annalen, vol. 63 (1907), p. 473.

† A special case, namely $\bar{T}(f) = a \cdot f$, has been considered by MARTY, Comptes Rendus, February 28, 1910, p. 515.

that is, the solutions form a biorthogonal system of the type T, and therefore the values of λ, for which there exist solutions, are real. The existence of real characteristic numbers λ is shown by means of the integral equation

$$(B) \qquad u(s) = \nu \int \bar{T}_s \bar{T}_t K(s,t) T[u(t)]dt,$$

which is of the type of the equation (14), for the kernel $\bar{T}_s \bar{T}_t K(s,t)$ is symmetric (Remark 3). It is evident that any solution of (A) for the value λ is a solution of (B) for $\nu = \lambda^2$. And unless $\bar{T}_s \bar{T}_t K(s,t)$ is of the form

$$\alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

there exists at least one value of ν which is real and positive and for which the equation (B) has a continuous solution u.

Construct the two functions

$$2u_1(s) = u(s) + \sqrt{\nu} \int T_s K(s,t)u(t)dt,$$

$$2u_2(s) = u(s) - \sqrt{\nu} \int T_s K(s,t)u(t)dt.$$

The continuous functions u_1 and u_2 are solutions of (A) for $\lambda = \sqrt{\nu}$ and $\lambda = -\sqrt{\nu}$ respectively.

Example 7. In certain cases the integral equation *

$$(\alpha) \qquad u(s) = \lambda \int K(s,t)u(t)dt + \mu \int H(s,t)u(t)dt,$$

where $K(s,t)$ and $H(s,t)$ are symmetric kernels continuous on S, λ and μ are parameters, and u the function to be determined, may be reduced to the class of integral equations considered in § 3.

Let λ_1 be that characteristic number of $K(s,t)$ which is smallest in absolute value. Then for any value of λ_0 such that

$$|\lambda_0| < |\lambda_1|,$$

there exists at least one value of μ for which the integral equation has a continuous solution u. For the maximum of

$$\left| \int\int K(s,t)f(s)f(t)ds\,dt \right|,$$

* MAX MASON, *Randwertaufgaben bei gewöhnlichen Differentialgleichungen, Dissertation*, Göttingen, 1903, p. 5.

for all continuous functions f such that

$$\int f^2 = 1,$$

is equal to the reciprocal of $|\lambda_1|$, and therefore the functional transformation

$$f(s) - \lambda_0 \int K(s, t) f(t)\, dt$$

satisfies all the conditions of § 1. Since λ_0 is not a characteristic number, it can easily be shown that the inverse transformation,

$$T(f) = f(s) - \lambda_0 \int k(s, t) f(t)\, dt,$$

where $k(s, t)$ is the reciprocal of $K(s, t)$, also satisfies the conditions of § 1. Operating with $T(f)$ on the integral equation (α), where $\lambda = \lambda_0$, we obtain

$$(\beta) \qquad u(s) = \mu \int T_t H(s, t) u(t)\, dt,$$

which is of the type of the equation (17), § 3. From the equation (β) we can pass back to the equation (α), and therefore the statement is proved.

Let $H(s, t)$ be a positive semi-definite kernel, and let λ be any number not equal to a characteristic number of $K(s, t)$; then the functional transformation

$$f(s) - \lambda \int k(s, t) f(t)\, dt$$

possesses all the properties imposed on the functional transformation $\bar{T}(f)$ of Example 6. Operating with this on the equation (α), we obtain an equation of the type discussed in Example 6.

Let λ take on a value equal to a characteristic number $\bar{\lambda}$ of $K(s, t)$ and let $H(s, t)$ be a positive semi-definite kernel which is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. The transformation

$$f(s) - \bar{\lambda} \int K(s, t) f(t)\, dt$$

defines an inverse transformation $\bar{T}(f)$ which has the property (b), § 1, and which is applicable to every function that is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. Operating with this $\bar{T}(f)$ on the equation (α), we obtain an equation of the same form as the one considered in Example 6, and it is not difficult to see that we again have the existence of values of μ besides the obvious one, $\mu = 0$.

The following is an example of the integral equation (α) in which the kernel $H(s, t)$ is not positive-definite, and to the value $\lambda = -1$ there corresponds no *real* value of μ.

$$u(s) = \lambda \int \psi(s)\psi(t)u(t)dt + \mu \int [h_1(s)h_1(t) - 2h_2(s)h_2(t)]u(t)dt,$$

where

$$\int \psi^2 = \int h_1^2 = \int h_2^2 = 1,$$

$$\int h_1 h_2 = 0, \qquad \int h_1 \psi = 0, \qquad \int h_2 \psi = 0.$$

March, 1910.

ON THE UNIFORM CONVERGENCE OF THE DEVELOPMENTS IN BESSEL FUNCTIONS*

BY

CHARLES N. MOORE

In many problems of mathematical physics we are led to the question of whether or not it is possible to represent an arbitrary function by a series of the form

$$A_1' J_\nu(\lambda_1 x) + A_2' J_\nu(\lambda_2 x) + \cdots, \tag{1}$$

where the A's are constants, $J_\nu(x)$ represents a Bessel function of order ν, and the λ's are the roots of the equation

$$l\lambda J_\nu'(\lambda) + hJ_\nu(\lambda) = 0, \tag{2}$$

in which h and l are any constants not both zero. In most cases we find that we have a solution of our problem if the A's can be so determined that:

(a) The series (1) converges uniformly to $f(x)$ throughout any closed interval lying in the interval $0 \leqq x \leqq 1$ and not including a point of discontinuity of $f(x)$, or, in some instances, the origin or the point $x = 1$, and

(b) The sum of the first n terms of the series remains finite for all values of n and all values of x lying in the interval $0 \leqq x \leqq 1$.

The ordinary formal method of determining the coefficients of an expansion of an arbitrary function in terms of normal functions, gives us the following value for A_n'

$$A_n' = \frac{\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx}{\int_0^1 x\,[J_\nu(\lambda_n x)]^2\,dx}, \tag{3}$$

where λ_n is the nth positive root of equation (2). This determination of the A's is found on investigation to give the desired expansion of $f(x)$ except in the case

$$l\nu + h = 0, \tag{4}$$

when an extra term is required in order that the series may have the proper value.

* Presented to the Society, October 30, 1909 and February 25, 1911.

That the case in which (4) is satisfied is really not an exceptional case at all, but only *seems* so on account of the notation employed, has been pointed out in a previous paper by the writer.* Moreover, as was also brought out in the previous paper, the appearance of an exceptional case can be avoided by introducing the notation

$$F_\nu(\lambda, x) \equiv \left(\frac{x}{2}\right)^\nu \sum_{i=0}^{\infty} \frac{(-1)^i (\lambda x)^{2i}}{2^{2i} i! \Gamma(\nu + i + 1)} \equiv \frac{1}{\lambda^\nu} J_\nu(\lambda x), \tag{5}$$

and considering the series

$$\sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 x f(x) F_\nu(\lambda_n, x) dx}{\int_0^1 x [F_\nu(\lambda_n, x)]^2 dx} = \sum_{n=1}^{\infty} A_n F_\nu(\lambda_n, x), \tag{6}$$

where the λ's are the successive roots, positive or zero, of the equation

$$\left[l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0, \dagger \tag{7}$$

arranged in increasing order of magnitude. This series is identical with the series (1) where the λ's are the successive positive roots of (2), except in the case where (4) is satisfied, and in that case only differs from it by having an extra term at the beginning. This extra term is finite and continuous, and hence in all questions involving merely convergence or the remaining finite of the sum of n terms of the series, we may deal with series (1), and it is only necessary to consider series (6) when we come to the question of the value of the development.

It has been shown by several writers‡ that the series (6) will, under very broad restrictions upon $f(x)$, converge uniformly to $f(x)$ throughout any closed interval, lying in the interval $0 < x < 1$ and not including a point of discontinuity of $f(x)$. The question of whether or not, under certain restrictions upon $f(x)$, the series (6) will converge to $f(0)$ at the origin, § and will converge uniformly to $f(x)$ in an interval including the origin and in an inter-

*Cf. these Transactions, vol. 10 (1909), p. 420. That the exceptional case was such only on account of the notation employed, and that it would cease to be exceptional if we adopted the proper notation, was pointed out to me by Professor BÔCHER, to whose valuable suggestions I owe many ideas that have been of great use in this as well as in the preceding investigation.

†The *positive* roots of this equation are obviously identical with the *positive* roots of equation (2).

‡Cf. DINI, *Serie di Fourier* (Pisa, 1880), pp. 246-269; KNESER, Mathematische Annalen, vol. 63 (1906-07), p. 505; HOBSON, Proceedings of the London Mathematical Society, ser. 2, vol. 7 (1909), p. 359.

§It is only in the case $\nu = 0$ that this question is anything but a trivial one, since for $\nu > 0$ every term of the series vanishes when $x = 0$ and the value of the series is obviously zero for this value of x.

val including the point $x = 1$ has not been settled in the papers referred to; neither has it been shown that, under suitable restrictions upon $f(x)$, the sum of n terms of the series will remain finite for all values of n and all values of x in the interval $0 \leqq x \leqq 1$. HILBERT has shown* for the particular case where $\nu = 0$ and the l of equation (7) is zero that if $f(x)$ vanishes for $x = 1$ and satisfies the rather stringent conditions of being continuous, together with its first and second derivatives, throughout the interval $0 \leqq x \leqq 1$, the series (6) will converge uniformly to $f(x)$ throughout the interval $0 \leqq x \leqq 1$.

For most of the applications to mathematical physics it is essential to know something about the behavior of the series (6) in the neighborhood of the origin, of the point $x = 1$, and of points of discontinuity of $f(x)$, when such exist, in order to be sure that we really have a solution of the physical problem we are discussing. In the majority of these cases, conditions like HILBERT's are too binding, and it is necessary to remove the restriction that the function should have no discontinuity; it is also necessary, even in the particular case that HILBERT treats, to determine the behavior of the series in the neighborhood of the point $x = 1$ when $f(1) \neq 0$.

The object of the present paper is to show that in case $f(x)$ is made up of a number of pieces in each of which $f(x)$ and its derivative $f'(x)$ are continuous, while $f''(x)$ exists, and is finite and integrable, the series (6) will possess the desired properties. These conditions on $f(x)$ are usually met with in the applications; in a later paper the writer proposes to discuss the uniform summability of the series in the neighborhood of $x = 0$ and $x = 1$, thus showing the existence of a solution of the physical problem under extremely broad conditions.†

LEMMA 1. *If in the interval $0 \leqq x \leqq 1$, the function $f(x)$ is continuous, save at a finite number of points at which it has a finite jump, if in each interval of continuity it has a first derivative that is continuous, save at a finite number of points at which it has a finite jump, and if finally in each interval of continuity of the first derivative the function has a second derivative that is finite and integrable, the coefficient A_n' of the general term of the series* (1) *may be written in the form*

$$(8)\qquad \sum_{m=1}^{m=k} M_m \frac{\cos\left[(n\pi+q)c_m - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} + Mf(1)\frac{\cos\left[(n\pi+q) - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} + \frac{\nu K f(0)}{\lambda_n} + r_n,$$

*Göttinger Nachrichten (1904), *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (*Zweite Mitteilung*), p. 229. Previously KNESER had discussed a special case slightly different from the one treated by HILBERT, using similar restrictions upon $f(x)$. (Cf. Archiv der Mathematik und Physik, 3rd ser., vol. 7 (1904), pp. 123–133.)

† Cf. the introduction to the paper in these Transactions referred to above.

where the M's and K are constants, the c's are the points of discontinuity of $f(x)$, and r_n is the general term of an absolutely convergent series.

We have *

$$(9 \qquad \int_0^1 x\,[J_\nu(\lambda_n x)]^2 dx = \frac{A^2}{2\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^2},$$

where A is a constant, and $\psi_1(\lambda_n)$ *is used to represent any function of* λ_n *that remains finite when* λ_n *increases indefinitely.* Hence it follows from (3) that if we subtract from A_n' the term

$$(10) \qquad \frac{2\lambda_n}{A^2}\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx,$$

we shall have left the term

$$(11) \qquad \frac{-\dfrac{2\psi_1(\lambda_n)}{A^2\lambda_n}\displaystyle\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx}{\dfrac{A^2}{2\lambda_n} + \dfrac{\psi_1(\lambda_n)}{\lambda_n^2}}.$$

We will show first that (11) is the general term of an absolutely convergent series. Then, that by subtracting from (10) quantities that are the general terms of absolutely convergent series, we can reduce it to the sum of the first three terms in (8). Our lemma will thus be established.

Let us use $\phi_m(x)$ to represent a function that is continuous in the interval $0 \leqq x \leqq c_{m+1}$, that has in this interval a first derivative that is continuous, save at a finite number of points at which it has a finite jump, and a second derivative that is finite and integrable, and furthermore is such that

$$(12) \qquad \phi_m(x) = f(x) \quad (c_m \leqq x \leqq c_{m+1};\ m = 0, 1, 2, \cdots, k),$$

where for the sake of uniformity we have set

$$c_0 = 0, \qquad c_{k+1} = 1.$$

Then we may write

$$(13) \qquad \int_0^1 x f(x) J_\nu(\lambda_n x)\,dx = \sum_{m=0}^{m=k}\int_0^{c_{m+1}} x\phi_m(x) J_\nu(\lambda_n x)\,dx - \sum_{m=1}^{m=k}\int_0^{c_m} x\phi_m(x) J_\nu(\lambda_n x)\,dx.$$

But if c is a constant lying in the interval $0 < x < 1$, and $\phi(x)$ is a function that is continuous in the interval $0 \leqq x \leqq c$ and has a first derivative that is continuous in this interval save at a finite number of points, we have,† if we set $\lambda_n x = y$ and integrate by parts, .

* Cf. equation (46) of the writer's article in these Transactions, just referred to.

† Cf. BYERLY, *An Elementary Treatise on Fourier's Series, etc.*, p. 223, formula (4).

$$\int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx = \frac{1}{\lambda_n^2}\int_0^{c\lambda_n} y^{\nu+1}J_\nu(y)\frac{1}{y^\nu}\phi\left(\frac{y}{\lambda_n}\right)dy$$

$$= \frac{1}{\lambda_n^2}\left[y^{\nu+1}J_{\nu+1}(y)\frac{1}{y^\nu}\phi\left(\frac{y}{\lambda_n}\right)\right]_0^{c\lambda_n}$$

$$-\frac{1}{\lambda_n^2}\int_0^{c\lambda_n} y^{\nu+1}J_{\nu+1}(y)\left\{-\frac{\nu}{y^{\nu+1}}\phi\left(\frac{y}{\lambda_n}\right)+\frac{1}{\lambda_n y^\nu}\phi'\left(\frac{y}{\lambda_n}\right)\right\}dy$$

$$(14)\qquad = \frac{1}{\lambda_n}cJ_{\nu+1}(c\lambda_n)\phi(c)+\frac{\nu}{\lambda_n^2}\int_0^{c\lambda_n} J_{\nu+1}(y)\phi\left(\frac{y}{\lambda_n}\right)dy$$

$$-\frac{1}{\lambda_n^3}\int_0^{c\lambda_n} yJ_{\nu+1}(y)\phi'\left(\frac{y}{\lambda_n}\right)dy$$

$$= \frac{c\phi(c)J_{\nu+1}(c\lambda_n)}{\lambda_n}+\frac{\nu}{\lambda_n}\int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx$$

$$-\frac{1}{\lambda_n}\int_0^c xJ_{\nu+1}(\lambda_n x)\phi'(x)\,dx.$$

Moreover, we have from the asymptotic expansion* of $J_{\nu+1}(x)$

$$(15)\qquad J_{\nu+1}(\lambda_n x) = \frac{\psi(\lambda_n x)}{\lambda_n^{\frac{1}{2}}x^{\frac{1}{2}}},$$

where *we use $\psi(x)$ to represent any function of x that is continuous for all values of $x>0$, remains finite for all such values of x, and approaches a finite limit when x approaches zero.*

From (14) and (15) we have

$$(16)\qquad \left|\int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx\right| < \frac{K_1}{\lambda_n^{\frac{3}{2}}},$$

where K_1 is a positive constant. Combining (16) and (13), we obtain

$$(17)\qquad \left|\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx\right| < \frac{K_2}{\lambda_n^{\frac{3}{2}}},$$

where K_2 is a positive constant. But for values of n so great that

$$\left|\frac{\psi_1(\lambda_n)}{\lambda_n^2}\right| < \frac{A^2}{4\lambda_n},$$

the quantity (11) is less in absolute value than

$$\frac{8\,|\psi_1(\lambda_n)|}{A^4}\cdot\left|\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx\right|,$$

*Cf. LIPSCHITZ, Crelle's Journal, vol. 56 (1859), pp. 193–196.

and hence, in view of (17), is less in absolute value than $K_3/\lambda_n^{\frac{3}{2}}$, where K_3 is a positive constant. Consequently, since a positive constant C exists such that *

$$\lambda_n > Cn, \tag{18}$$

it follows that (11) is the general term of an absolutely convergent series.

It remains to consider the term (10) which was subtracted from A_n' to produce (11). We must first, however, discuss the last two terms on the right hand side of equation (14).

The asymptotic expansion of $J_{\nu+1}(x)$ gives

$$J_{\nu+1}(\lambda_n x) = \sqrt{\frac{2}{\pi\lambda_n x}} \cos\left(\lambda_n x - \frac{2\nu+3}{4}\pi\right) + \frac{\psi(\lambda_n x)}{\lambda_n^{\frac{3}{2}} x^{\frac{3}{2}}}. \tag{19}$$

Substituting this value in the integral of the third term on the right hand side of (14), we obtain

$$\begin{aligned}\int_0^c x\phi'(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{A}{\sqrt{\lambda_n}} \int_0^c \sqrt{x}\phi'(x)\cos(\lambda_n x - \alpha)\,dx \\ + \frac{1}{\lambda_n^{\frac{3}{2}}} \int_0^c \frac{\phi'(x)}{\sqrt{x}} \psi(\lambda_n x)\,dx,\end{aligned} \tag{20}$$

where for simplicity we have used A and α in place of $\sqrt{2/\pi}$ and $(2\nu+3)\pi/4$.

If $\phi(x)$ is a function whose first derivative is continuous in the interval $0 \leqq x \leqq c$, except at a finite number of points at which it has a finite jump, and whose second derivative is finite and integrable there, we have from an integration by parts

$$\begin{aligned}\int_0^c \sqrt{x}\phi'(x)\cos(\lambda_n x - \alpha)\,dx = \frac{1}{\lambda_n} \sum_{i=1}^{i=r+1} \left[\sqrt{k_i}\phi'(k_i - 0)\sin(\lambda_n k_i - \alpha)\right. \\ \left. - \sqrt{k_{i-1}}\phi'(k_{i-1} + 0)\sin(\lambda_n k_{i-1} - \alpha)\right] \\ - \frac{1}{\lambda_n}\int_0^c \left[\sqrt{x}\phi''(x) + \frac{\phi'(x)}{2\sqrt{x}}\right]\sin(\lambda_n x - \alpha)\,dx,\end{aligned} \tag{21}$$

where $k_1, k_2, \cdots, k_r$ are the points of discontinuity of $\phi'(x)$, and for the sake of uniformity we have set

$$k_0 = 0, \qquad k_{r+1} = c.$$

Combining (20) and (21), we have

$$\int_0^c x\phi'(x) J_{\nu+1}(\lambda_n x)\,dx = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}. \tag{22}$$

If we set

$$F(x) = \frac{\phi(x) - \phi(0)}{x}, \tag{23}$$

* Cf. equation (23) of the writer's article in these Transactions, referred to above.

we obtain for the integral in the second term on the right hand side of (14)

$$(24)\quad \int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx = \phi(0)\int_0^c J_{\nu+1}(\lambda_n x)\,dx + \int_0^c xF(x)J_{\nu+1}(\lambda_n x)\,dx.$$

If we substitute in the second term on the right hand side of (24) the value of $J_{\nu+1}(\lambda_n x)$ given by (19), we get

$$(25)\quad \int_0^c xF(x)J_{\nu+1}(\lambda_n x)\,dx = \frac{A}{\sqrt{\lambda_n}}\int_0^c \sqrt{x}\cos(\lambda_n x - \alpha)F(x)\,dx + \frac{1}{\lambda_n^{\frac{3}{2}}}\int_0^c \frac{\psi(\lambda_n x)}{\sqrt{x}}F(x)\,dx,$$

where again, for the sake of simplicity, we have introduced A and α in place of $\sqrt{2/\pi}$ and $(2\nu+3)\pi/4$. By means of an integration by parts, the integral in the first term on the right hand side of (25) reduces to the form

$$(26)\quad \frac{\sin(\lambda_n c - \alpha)}{\lambda_n}\sqrt{c}\,F(c) - \frac{1}{\lambda_n}\int_0^c \left[\sqrt{x}F'(x) + \frac{F(x)}{2\sqrt{x}}\right]\sin(\lambda_n x - \alpha)\,dx.$$

Substituting this expression in (25), we obtain for the second term on the right hand side of (24)

$$(27)\quad \int_0^c xF(x)J_{\nu+1}(\lambda_n x)\,dx = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.$$

If we set $\lambda_n x = y$, we get for the integral in the first term on the right hand side of (24)

$$(28)\quad \int_0^c J_{\nu+1}(\lambda_n x)\,dx = \frac{1}{\lambda_n}\int_0^{c\lambda_n} J_{\nu+1}(y)\,dy,$$

where the integral on the right hand side may be written

$$(29)\quad \int_0^\infty J_{\nu+1}(y)\,dy - \int_{c\lambda_n}^\infty J_{\nu+1}(y)\,dy,$$

since these integrals converge.

Making use of (19), we obtain for the second term in (29)

$$(30)\quad \int_{c\lambda_n}^\infty J_{\nu+1}(y)\,dy = A\int_{c\lambda_n}^\infty \frac{\cos(y-\alpha)}{\sqrt{y}}\,dy + \int_{c\lambda_n}^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\,dy,$$

where this reduction is legitimate since the two integrals on the right hand side of (30) both converge. For the second of these integrals we have

$$(31)\quad \left|\int_{c\lambda_n}^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\,dy\right| < N\int_{c\lambda_n}^\infty \frac{dy}{y^{\frac{3}{2}}} = \frac{N}{c^{\frac{1}{2}}\lambda_n^{\frac{1}{2}}},$$

where N is a positive constant. For the first integral on the right hand side of (30) we have from an integration by parts

$$(32)\quad \int_{c\lambda_n}^{\infty} \frac{\cos(y-\alpha)}{\sqrt{y}}\,dy = \left[\frac{\sin(y-\alpha)}{\sqrt{y}}\right]_{c\lambda_n}^{\infty} + \frac{1}{2}\int_{c\lambda_n}^{\infty} \frac{\sin(y-\alpha)}{y^{\frac{3}{2}}}\,dy = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{1}{2}}}.$$

Combining (31) and (32) with (30), we obtain

$$(33)\quad \int_{c\lambda_n}^{\infty} J_{\nu+1}(y)\,dy = \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{1}{2}}}.$$

Substituting (33) in (29), and remembering that (29) is only another way of writing the integral on the right hand side of (28), we get

$$(34)\quad \int_0^c J_{\nu+1}(\lambda_n x)\,dx = \frac{K'}{\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}},$$

where for the sake of simplicity we have used K' in place of the first term in (29).

Combining (24), (27), and (34), we obtain

$$(35)\quad \int_0^c J_{\nu+1}(\lambda_n x)\phi(x)\,dx = \frac{K'\phi(0)}{\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.$$

From (14), (22), and (35), we get

$$(36)\quad \int_0^c x\phi(x)J_\nu(\lambda_n x)\,dx = \frac{c\phi(c)J_{\nu+1}(c\lambda_n)}{\lambda_n} + \frac{\nu K'\phi(0)}{\lambda_n^2} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.$$

Finally from (13) and (36), we obtain

$$(37)\quad \begin{aligned}\int_0^1 xf(x)J_\nu(\lambda_n x)\,dx = &\sum_{m=0}^{m=k} \frac{c_{m+1}\phi_m(c_{m+1})J_{\nu+1}(c_{m+1}\lambda_n)}{\lambda_n} \\ &- \sum_{m=1}^{m=k} \frac{c_m\phi_m(c_m)J_{\nu+1}(c_m\lambda_n)}{\lambda_n} + \frac{\nu K'f(0)}{\lambda_n^2} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}}}.\end{aligned}$$

It follows from (37) that by subtracting from (10) a quantity that is the general term of an absolutely convergent series, we may reduce it to the form

$$(38)\quad \frac{2}{A^2}\sum_{m=0}^{m=k} c_{m+1}\phi_m(c_{m+1})J_{\nu+1}(c_{m+1}\lambda_n) - \frac{2}{A^2}\sum_{m=1}^{m=k} c_m\phi_m(c_m)J_{\nu+1}(c_m\lambda_n) + \frac{\nu Kf(0)}{\lambda_n},$$

where we have replaced $2K'/A^2$ by K. If, now, we can show that by subtracting from (38) a quantity that is the general term of an absolutely convergent series we may reduce it to the sum of the first three terms in (8), our lemma will be proved.

We have [loc. cit., equation (58)]

$$(39)\quad \cos\left(\lambda_n c - \frac{(2\nu+3)\pi}{4}\right) = \cos\left[(n\pi+q)c - \frac{(2\nu+3)\pi}{4}\right] + \frac{\psi_1(\lambda_n)}{n}.$$

Combining (39) with (19) and (18), we obtain

$$(40)\qquad J_{\nu+1}(\lambda_n c) = \sqrt{\frac{2}{\pi\lambda_n c}}\cos\left[(n\pi+q)c - \frac{(2\nu+3)\pi}{4}\right] + \frac{\psi_1(\lambda_n)}{n^{\frac{3}{2}}}.$$

But since [loc. cit. equation (57)]

$$(41)\qquad \lambda_n = n\pi + q + \frac{\psi_1(\lambda_n)}{n},$$

we have by (18)

$$(42)\qquad \frac{1}{\sqrt{\lambda_n}} - \frac{1}{\sqrt{n\pi}} = \frac{n\pi - \lambda_n}{\sqrt{\lambda_n}\, n\pi(\sqrt{\lambda_n} + \sqrt{n\pi})} = \frac{\psi_1(n)}{n^{\frac{3}{2}}}.$$

Combining (40) and (42), we obtain

$$(43)\qquad J_{\nu+1}(\lambda_n c) = \frac{\sqrt{2}\cos\left[(n\pi+q)c - \dfrac{2\nu+3}{4}\pi\right]}{\pi\sqrt{nc}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}}.$$

It follows at once from (43) that by subtracting from (38) a quantity that is the general term of an absolutely convergent series, we can reduce it to the sum of the first three terms in (8). Our lemma is therefore established.

Before beginning the proof of the next lemma, we shall state a few formulæ connected with Bessel functions, which will be useful in the course of that proof.

We have [loc. cit. equation (58)]

$$(44)\qquad \cos(\lambda_n x - \alpha) = \cos(n\pi x + qx - \alpha) + \frac{\psi(x, \lambda_n)}{n},$$

where λ_n is the nth positive root of equation (2) and

$$(45)\qquad |\psi(x, \lambda_n)| < K_1 \qquad (0 \leqq x \leqq 1;\ n = 1, 2, 3, \cdots),$$

K_1 being a positive constant.

From the asymptotic expansion * we have

$$(46)\qquad J_\nu(\lambda_n x) = \frac{A}{\sqrt{\lambda_n x}}\cos\left(\lambda_n x - \frac{(2\nu+1)\pi}{4}\right) + \frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}},$$

where

$$(47)\qquad |\psi(x)| < K_2, \qquad |\psi'(x)| < K_3 \qquad (x > 0),$$

K_2 and K_3 being positive constants.

* Cf. LIPSCHITZ, loc. cit. That the second inequality in (47) holds may be seen by differentiating the expression of the form (47) for $J_\nu(x)$ and comparing the result with the value of $J'_\nu(x)$ obtained by combining the equation

$$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x}J_\nu(x)$$

with the expression of the form (46) for $J_\nu(x)$ and the expression of the form (19) for $J_{\nu+1}(x)$.

LEMMA 2. *The quantity*

$$\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{48}$$

where c is any positive constant $\leqq 1$, is the general term of a series that is uniformly convergent in any interval

$$0 \leqq x \leqq x_0 < c. \tag{49}$$

The quantity (48) may be written in the form

$$\cos\left(qc-\frac{2\nu+3}{4}\pi\right)\frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}} - \sin\left(qc-\frac{2\nu+3}{4}\pi\right)\frac{\sin n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{50}$$

and hence if we can show that each of the quantities

$$\frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}}, \qquad \frac{\sin n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}}, \tag{51}$$

is the general term of a series that is uniformly convergent in the interval (49), our lemma will be proved.

Let us consider the series whose general term is the first expression in (51); the proof for the series whose general term is the second expression in (51) is entirely analogous. Since $0 < c < 2$, the series that we are discussing converges for $x = 0$;* it remains to be shown that it converges uniformly in the interval $0 < x \leqq x_0 < c$.

Given δ, positive and arbitrarily small, let us choose m such that †

$$\left|\sum_{n=p}^{n=q}\frac{\sin}{\cos}\frac{(n\pi z)}{\sqrt{n}}\right| < \eta = \frac{\delta}{1+C_2+C_3} \qquad \left(\begin{matrix} c-x_0 \leqq z \leqq c+x_0 \\ q > p \geqq m\end{matrix}\right), \tag{52}$$

$$\sum_{n=p}^{n=q}\frac{1}{n^{\frac{3}{2}}} < \eta \qquad (q > p \geqq m), \tag{53}$$

where

$$C_2 = \frac{A}{\sqrt{k}}(K_1+4), \qquad C_3 = \frac{2^{\frac{3}{2}}N^{\frac{1}{2}}(3CK_3k+2K_2N)+6K_2C^{\frac{1}{2}}}{3C^{\frac{1}{2}}k^{\frac{3}{2}}}, \tag{54}$$

the symbols A, C, K_1, K_2, K_3 representing the same quantities as in equations

* For $\nu > 0$ all the terms of the series are zero, and its convergence is obvious; for $\nu = 0$, cf. SCHLÖMILCH, *Algebraische Analysis*, 3d ed., 1862, § 31, also BÔCHER, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 109–111.

† That the inequality (52) holds for sufficiently large values of m is evident from a reference to page 111 of the article by BÔCHER, quoted in the previous footnote.

(46), (18), (45), and (47); k is the smallest positive root of $J_\nu'(x) = 0$, and N is a positive constant such that *

$$\lambda_n < Nn. \tag{55}$$

Let s be the greatest integer such that $\lambda_s x < k$, and μ and ν any two integers such that

$$m \leqq \mu \leqq \nu.$$

Three cases must be considered:

$$(A)\quad \mu \leqq s < \nu, \qquad (B)\quad \nu \leqq s, \qquad (C)\quad \mu > s.$$

Beginning with (A), we write

$$\sum_{n=\mu}^{n=\nu} \frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}} = \sum_{n=\mu}^{n=s} + \sum_{n=s+1}^{n=\nu} = R_1 + R_2. \tag{56}$$

From (52) and a lemma due to ABEL,† we have

$$|R_1| < \eta \qquad (0 < x \leqq x_0) \tag{57}$$

For R_2 we have from (46)‡

$$R_2 = A \sum_{s+1}^{\nu} \frac{\cos n\pi c \cos(\lambda_n x - \alpha)}{\sqrt{n\lambda_n x}} + \sum_{s+1}^{\nu} \frac{\cos n\pi c\, \psi(\lambda_n x)}{\sqrt{n\lambda_n^3 x^3}}. \tag{58}$$

The general term of the first series on the right hand side of (58) may, in view of (44), be written in the form

$$\frac{\cos n\pi c \cos(n\pi x + qx - \alpha)}{\sqrt{n\lambda_n x}} + \frac{\cos n\pi c\, \psi(x, \lambda_n)}{n\sqrt{n\lambda_n x}}. \tag{59}$$

From (45), (53), and the fact that s was so chosen that $\lambda_{s+1} x \geqq k$, we have for the series whose general term is the second term in (59)

$$\left| \sum_{n=s+1}^{n=\nu} \frac{\cos n\pi c\, \psi(x, \lambda_n)}{n\sqrt{n\lambda_n x}} \right| < \sum_{n=s+1}^{n=\nu} \frac{K_1}{\sqrt{k} n^{\frac{3}{2}}} < \frac{K_1}{\sqrt{k}} \eta \qquad (0 < x \leqq x_0). \tag{60}$$

The first term in (59) reduces to the form

$$\begin{aligned} &\frac{\cos n\pi c \cos n\pi x}{\sqrt{n\lambda_n x}} \cos(qx - \alpha) - \frac{\cos n\pi c \sin n\pi x}{\sqrt{n\lambda_n x}} \sin(qx - \alpha) \\ &\quad = \frac{\cos(qx - \alpha)}{2} \left[\frac{\cos n\pi(c + x) + \cos n\pi(c - x)}{\sqrt{n\lambda_n x}} \right] \\ &\quad\quad - \frac{\sin(qx - \alpha)}{2} \left[\frac{\sin n\pi(c + x) - \sin n\pi(c - x)}{\sqrt{n\lambda_n x}} \right]. \end{aligned} \tag{61}$$

* That such a positive constant exists, is easily inferred from equation (41).

† Cf. TANNERY, *Fonctions d'une variable*, § 70. Abel's lemma applies by virtue of the fact that $J_\nu(\lambda_n x)$ steadily decreases or steadily increases with increasing n as long as $n \leqq s$.

‡ For the sake of convenience in writing we have replaced $(2\nu + 1)\pi/4$ by α.

But we have from (52)

$$(62)\qquad \begin{aligned} &\left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n}}\right|<\eta, \qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c-x)}{\sqrt{n}}\right|<\eta,\\ &\left|\sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c+x)}{\sqrt{n}}\right|<\eta, \qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c-x)}{\sqrt{n}}\right|<\eta, \end{aligned}$$

when x varies in the interval $0<x\leqq x_0$. Hence, from the lemma of Abel referred to above,

$$(63)\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n\lambda_n x}}\right|=\frac{1}{\sqrt{\lambda_{s+1}x}}\left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c+x)}{\sqrt{n}}\cdot\frac{\sqrt{\lambda_{s+1}}}{\sqrt{\lambda_n}}\right|<\frac{\eta}{\sqrt{k}} \qquad (0<x\leqq x_0).$$

By the use of the other inequalities in (62) and Abel's lemma, we can obtain inequalities analogous to (63) for each of the sums

$$\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi(c-x)}{\sqrt{n\lambda_n x}}, \qquad \sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c+x)}{\sqrt{n\lambda_n x}}, \qquad \sum_{n=s+1}^{n=\nu}\frac{\sin n\pi(c-x)}{\sqrt{n\lambda_n x}}.$$

Combining these inequalities and (63), and taking note of the fact that (61) is another way of writing the first term in (59), we obtain

$$(64)\qquad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c\cos(n\pi x+qx-\alpha)}{\sqrt{n\lambda_n x}}\right|<\frac{4\eta}{\sqrt{k}} \qquad (0<x\leqq x_0).$$

Since (59) is another way of writing the general term of the first series on the right hand side of (58), we have from (60), (64), and (59)

$$(65)\qquad \left|A\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c\cos(\lambda_n x-\alpha)}{\sqrt{n\lambda_n x}}\right|<\frac{A(K_1+4)\eta}{\sqrt{k}}=C_2\eta \qquad (0<x\leqq x_0).$$

We consider next the series that forms the second term on the right hand side of (58). If we set

$$(66)\qquad \epsilon_n=-\sum_{r=n}^{r=\infty}\frac{\cos r\pi c}{\sqrt{r}},$$

we have for this series*

$$(67)\qquad \begin{aligned} \sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c}{\sqrt{n}}\cdot\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{1}{4}}} &= -\epsilon_{s+1}\frac{\psi(\lambda_{s+1}x)}{(\lambda_{s+1}x)^{\frac{1}{4}}}\\ &\quad+\sum_{n=s+1}^{n=\nu}\left(\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{1}{4}}}-\frac{\psi(\lambda_{n+1}x)}{(\lambda_{n+1}x)^{\frac{1}{4}}}\right)\epsilon_{n+1}+\epsilon_{\nu+1}\frac{\psi(\lambda_{\nu+1}x)}{(\lambda_{\nu+1}x)^{\frac{1}{4}}}. \end{aligned}$$

But from the Law of the Mean

$$(68)\qquad \frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{1}{4}}}-\frac{\psi(\lambda_{n+1}x)}{(\lambda_{n+1}x)^{\frac{1}{4}}}=\frac{\psi'(\xi_n x)}{\xi_n^{\frac{1}{4}}x^{\frac{1}{4}}}-\frac{3\psi(\xi_n x)}{2\xi_n^{\frac{5}{4}}x^{\frac{1}{4}}} \qquad (\lambda_n<\xi_n<\lambda_{n+1}).$$

* Cf. equation (26) of the author's article in these Transactions, vol. 8 (1907), p. 299.

Moreover, in view of (47), (18), and (55), we have

$$(69)\quad \left|\sum_{n=s+1}^{n=\nu}\frac{\psi'(\xi_n x)}{\xi_n^{\frac{3}{2}}x^{\frac{1}{2}}}\right| < \sum_{n=s+1}^{n=\nu}\frac{K_3}{\lambda_n^{\frac{3}{2}}x^{\frac{1}{2}}} < \frac{K_3}{C^{\frac{3}{2}}x^{\frac{1}{2}}}\sum_{n=s+1}^{n=\nu}\frac{1}{n^{\frac{3}{2}}} < \frac{K_3}{C^{\frac{3}{2}}x^{\frac{1}{2}}}\int_{\epsilon}^{\nu}\frac{dx}{x^{\frac{3}{2}}} < \frac{2K_3}{C^{\frac{3}{2}}(sx)^{\frac{1}{2}}}$$
$$= \frac{2K_3}{C^{\frac{3}{2}}[(s+1)x]^{\frac{1}{2}}}\cdot\left(\frac{s+1}{s}\right)^{\frac{1}{2}} < \frac{2^{\frac{3}{2}}K_3N^{\frac{1}{2}}}{C^{\frac{3}{2}}(\lambda_{s+1}x)^{\frac{1}{2}}} < \frac{2^{\frac{3}{2}}K_3N^{\frac{1}{2}}}{C^{\frac{3}{2}}k^{\frac{1}{2}}} \qquad (x>0),$$

and

$$(70)\quad \left|\sum_{n=s+1}^{n=\nu}\frac{\psi(\xi_n x)}{\xi_n^{\frac{3}{2}}x^{\frac{3}{2}}}\right| < \sum_{s+1}^{\nu}\frac{K_2}{\lambda_n^{\frac{3}{2}}x^{\frac{3}{2}}} < \frac{K_2}{C^{\frac{3}{2}}x^{\frac{3}{2}}}\sum_{s+1}^{\nu}\frac{1}{n^{\frac{3}{2}}} < \frac{K_2}{C^{\frac{3}{2}}x^{\frac{3}{2}}}\int_{\epsilon}^{\nu}\frac{dx}{x^{\frac{3}{2}}}$$
$$< \frac{2K_2}{3C^{\frac{3}{2}}(sx)^{\frac{3}{2}}} < \frac{2^{\frac{3}{2}}K_2N^{\frac{3}{2}}}{3C^{\frac{3}{2}}k^{\frac{3}{2}}} \qquad (x>0).$$

From (66) and (52) we have

$$(71)\quad |\epsilon_n| \leqq \eta \qquad (n \geqq m).$$

Combining (67) with (68), (69), (70), (71), and (47), we obtain

$$(72)\quad \left|\sum_{n=s+1}^{n=\nu}\frac{\cos n\pi c}{\sqrt{n}}\cdot\frac{\psi(\lambda_n x)}{(\lambda_n x)^{\frac{3}{2}}}\right| < \frac{2^{\frac{3}{2}}N^{\frac{1}{2}}(3CK_3k+2K_2N)+6K_2C^{\frac{3}{2}}}{3C^{\frac{3}{2}}k^{\frac{3}{2}}}\eta = C_3\eta \qquad (x>0).$$

From (58), (65), and (72) we have

$$(73)\quad |R_2| < (C_2+C_3)\eta \qquad (0<x\leqq x_0),$$

which, combined with (56) and (57), gives

$$(74)\quad \left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}}\right| < (1+C_2+C_3)\eta = \delta \qquad (0<x\leqq x_0).$$

By a discussion precisely analogous to that of R_1 in Case (A), we get for Case (B)

$$(75)\quad \left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c\, J_\nu\ \lambda_n x)}{\sqrt{n}}\right| < \eta < \delta \qquad (0<x\leqq x_0),$$

and by a discussion precisely analogous to that of R_2 in Case (A), we get for Case (C)

$$(76)\quad \left|\sum_{n=\mu}^{n=\nu}\frac{\cos n\pi c\, J_\nu(\lambda_n x)}{\sqrt{n}}\right| < (C_2+C_3)\eta < \delta \qquad (0<x\leqq x_0).$$

Combining (74), (75), and (76), we see that the series whose general term is the first expression in (51) converges uniformly for all values of x lying in the interval $0 < x \leqq x_0$. Since the series is also convergent for $x = 0$, it follows that it is uniformly convergent in the interval $0 \leqq x \leqq x_0$. As we have pointed out before, the discussion of the series whose general term is the second expression in (51), is entirely analogous, and hence our lemma is to be regarded as completely established.

LEMMA 3. *If the quantity* (48) *is written in the form*

$$(77)\quad -\frac{A}{2\sqrt{\pi x}}\left[\cos\{q(x+c)-\nu\pi\}\frac{\cos n\pi(x+c)}{n}-\sin\{q(x+c)-\nu\pi\}\frac{\sin n\pi(x+c)}{n}\right.$$
$$\left.+\cos q(x-c)\frac{\sin n\pi(x-c)}{n}+\sin q(x-c)\frac{\cos n\pi(x-c)}{n}\right]$$
$$+v_n(x),$$

$v_n(x)$ will be the general term of a series that is uniformly convergent in any interval

$$(78)\qquad 0<x_0\leqq x\leqq 1.$$

By means of (46), (48) reduces to the form

$$(79)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left(\lambda_n x-\frac{2\nu+1}{4}\pi\right)}{\sqrt{n\lambda_n x}}$$
$$+\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi(\lambda_n x)}{\sqrt{n(\lambda_n x)^3}}.$$

The second term of (79) is obviously the general term of a series that is uniformly convergent in the interval (78); if then, by subtracting from the first term a number of quantities that have the same property, we can reduce it to an expression which is equal to the first term in (77), our lemma will be established.

From (44) we see that the first term in (79) can be written in the form

$$(80)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left[(n\pi+q)x-\frac{2\nu+1}{4}\pi\right]}{\sqrt{n\lambda_n x}}$$
$$+A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi(x,\lambda_n)}{n\sqrt{n\lambda_n x}}.$$

Since the second term of (80) is obviously the general term of a series that is uniformly convergent in the interval (78), it only remains to consider the first term. By means of (42) it can be reduced to the form

$$(81)\quad A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\cos\left[(n\pi+q)x-\frac{2\nu+1}{4}\pi\right]}{n\sqrt{\pi x}}$$
$$+A\frac{\cos\left[(n\pi+q)c-\frac{2\nu+3}{4}\pi\right]\psi_1(n)}{n^{\frac{3}{2}}\sqrt{nx}}.$$

Here again the second term is obviously the general term of a series that is uniformly convergent in the interval (78); if we can show further that the first term is equal to the first term of (77), our lemma will be proved.

By means of a well known trigonometric formula the first term of (81) may be reduced to the form

$$A\frac{\cos\left[(n\pi+q)(x+c)-(\nu+1)\pi\right]+\cos\left[(n\pi+q)(x-c)+\frac{\pi}{2}\right]}{2n\sqrt{\pi x}}$$

$$=-A\frac{\cos\left[(n\pi+q)(x+c)-\nu\pi\right]+\sin(n\pi+q)(x-c)}{2n\sqrt{\pi x}} \tag{82}$$

$$=-\frac{A}{2\sqrt{\pi x}}\left[\begin{array}{l}\cos\{q(x+c)-\nu\pi\}\dfrac{\cos n\pi(x+c)}{n}-\sin\{q(x+c)-\nu\pi\}\dfrac{\sin n\pi(x+c)}{n}\\ \\ \quad+\cos q(x-c)\dfrac{\sin n\pi(x-c)}{n}+\sin q(x-c)\dfrac{\cos n\pi(x-c)}{n}\end{array}\right].$$

The right hand side of (82) is identical with the first term of (77), and consequently our lemma is completely established.

LEMMA 4. *The quantity* (48) *is the general term of a series for which the sum of the first* n *terms remains finite for all values of* n *and all values of* x *lying in the interval* $0 \leqq x \leqq 1$.

That the sum of the first n terms of this series remains finite for all values of n and all values of x lying in the interval $0 \leqq x \leqq x_0 < c$, is an immediate consequence of Lemma 2. It remains to be shown that this sum remains finite for all values of n and all values of x lying in the interval (78).

We know from Lemma 3 that when the expression (48) is written in the form (77), $v_n(x)$ is the general term of a series that is uniformly convergent in the interval (78) and for which therefore the sum of the first n terms remains finite for all values of n and all values of x in the interval (78). It only remains to be shown then that the first term of (77) is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x lying in the interval (78). We will prove this by showing that each term of the quantity in brackets in (77) is the general term of such a series.

The series

$$\sum_{n=1}^{\infty}\frac{\sin n\pi z}{n} \tag{83}$$

is the Fourier's development of the periodic function

$$\phi(z)=\frac{\pi-z}{2} \qquad 0<z<2\pi,$$

$$\phi(z+2\pi)=\phi(z) \qquad z\neq 0, \pm 2\pi, \pm 4\pi, \cdots$$

and consequently, from a well known property of the Fourier's developments, the sum of the first n terms of this series remains finite for all values of n and all values of z. Hence the series whose general terms are the second and third terms of the expression in brackets in (77) have the property we desire to establish, and it only remains to examine the series whose general terms are the first and fourth terms of this expression.

If we represent by $s_n(z)$ the sum of the first n terms of the series

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n}, \tag{84}$$

we have * for $\sin(z/2) \neq 0$

$$s_n(z) = \frac{-\sin\frac{z}{2} + \sum_{m=1}^{m=n-1}\left(\frac{1}{m} - \frac{1}{m+1}\right)\sin\frac{2m+1}{2}z + \frac{1}{n}\sin\frac{2n+1}{2}z}{2\sin\frac{z}{2}}.$$

Consequently

$$|s_n(z)| < \frac{1 + \sum_{m=1}^{m=n-1}\left(\frac{1}{m} - \frac{1}{m+1}\right) + \frac{1}{n}}{2\left|\sin\frac{z}{2}\right|} = \frac{1}{\left|\sin\frac{z}{2}\right|}, \tag{85}$$

and hence we have for the sum of the first n terms of the series whose general term is the fourth term of the expression in brackets in (77)

$$|s_n(x)| < \frac{|\sin q(x-c)|}{|\sin \frac{1}{2}\pi(x-c)|}.$$

It follows from the last inequality that the sum of the first n terms of this series remains finite for all values of n and all values of x lying in the interval (78).

We have finally to consider the series whose general term is the first term of the expression in brackets in (77). In view of (85) we have for the first n terms of this series

$$|s_n(x)| < \frac{|\cos\{q(x+c) - \nu\pi\}|}{|\sin\frac{1}{2}\pi(x+c)|}. \tag{86}$$

In order to assure ourselves that $s_n(x)$ remains finite for all values of n and *all* values of x lying in the interval (78) we must examine the values that q may take on. We have †

$$q = \frac{\pi}{2} + \frac{2\nu+1}{4}\pi + k_1\pi \quad \text{or} \quad q = \frac{2\nu+1}{4}\pi + k_2\pi, \tag{87}$$

* Cf. the writer's paper, these Transactions, vol. 10 (1909), p. 408, equation (66).

† Loc. cit., p. 415. The α in these values of q was used, for the sake of simplicity, in place of the quantity $(2\nu+1)\pi/4$.

where k_1 and k_2 are positive or negative integers or zero. Consequently, even for $c = 1$, the expression on the right hand side of (86) remains finite for all values of x lying in the interval (78), and hence the same is true of $s_n(x)$.

We have now shown that each term of the expression in brackets in (77) is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x in the interval (78). Hence the whole first term of (77) is the general term of such a series and our lemma is proved.

LEMMA 5. *The expression* (48) *is, for the case* $c < 1$, *the general term of a series that is uniformly convergent in any interval*

$$c < x_1 \leqq x \leqq 1; \tag{88}$$

for the case $c = 1$, *it is the general term of a series that is uniformly convergent in any interval*

$$0 < x_1 \leqq x \leqq 1, \tag{89}$$

provided the l *of equation* (2) *is not zero.*

Let us consider first the case where $c < 1$. We know from Lemma 3 that when (48) is written in the form (77), $v_n(x)$ is the general term of a series that is uniformly convergent in the interval (88). It remains to be shown that the first term of (77) is also the general term of a series that is uniformly convergent in this interval.

The series

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos nz}{n}$$

converge uniformly * throughout the interval

$$\delta_1 \leqq z \leqq 2\pi - \delta_2,$$

where δ_1 and δ_2 are fixed positive quantities, which may be taken as small as we please. Hence each term of the quantity in brackets in (77) is the general term of a series that is uniformly convergent in the interval (88). Therefore the whole first term is the general term of such a series, and our lemma is proved for the case where $c < 1$.

We turn now to the case $c = 1$, $l \neq 0$. As in the previous case, we may, by the use of Lemma 3, reduce the proof of our lemma to the proof of the fact that the first term of (77) is the general term of a series that is uniformly convergent in the interval in question. We will now establish this fact by showing that for the values that c and l have in the case we are considering, this term is identically zero. Thus our lemma will be completely proved.

*Cf. BÔCHER, loc. cit., p. 111.

Since the l of equation (2) is not zero, we have for q [cf. the footnote to (87)]

$$q = k_2\pi + \frac{2\nu+1}{4}\pi. \tag{90}$$

Moreover, we saw in the course of the proof of Lemma 3 that the first term in (81) was identical with the first term in (77). If we substitute in the first term of (81) the value $c = 1$ and the value of q given by (90), we find that the first factor in the numerator of this term vanishes identically. Hence the first term of (77) vanishes for the values of l and c in question and, as pointed out before, our lemma is proved.

LEMMA 6. *The expression*

$$\frac{1}{\lambda_n} J_\nu(\lambda_n x), \tag{91}$$

where $\nu > 0$, is the general term of a series for which the sum of the first n terms remains finite for all values of n and all values of x lying in the interval

$$0 \leqq x \leqq 1. \tag{92}$$

The theorem is obvious for $x = 0$, since all the terms of the series vanish in that case. Hence it only remains to establish it for the interval

$$0 < x \leqq 1. \tag{93}$$

Let α_1 and α_2 be the smallest positive roots of

$$\frac{d}{dx}\left\{\frac{J_\nu(x)}{x}\right\} = 0, \tag{94}$$

and $J_\nu(x)$ respectively, and let k be the smaller of the two quantities α_1, α_2. Now choose a positive integer q such that $\lambda_q > k$ and

$$\frac{\lambda_{m+1}}{\lambda_{m-1}} < 2 \qquad (m \geqq q). \tag{95}$$

From (15) we have

$$|J_\nu(\lambda_n x)| < \frac{N}{\lambda_n^{\frac{1}{2}} x^{\frac{1}{2}}}, \tag{96}$$

where N is a positive constant. Hence for values of x lying in the interval

$$\frac{k}{\lambda_q} \leqq x \leqq 1, \tag{97}$$

the series whose general term is (91) is uniformly convergent, and consequently the sum of the first n terms of this series remains finite for all values of n and all values of x lying in this interval. If we can show further that the sum of

the first n terms of the series remains finite for all values of n and all values of x lying in the interval

$$0 < x < \frac{k}{\lambda_q}, \tag{98}$$

our lemma will be completely established.

Let us consider, then, any value of x lying in the interval (98), and let p be the largest positive integer such that $\lambda_p x < k$. There will be two cases

$$(A)\quad p \leqq n, \qquad (B)\quad p > n.$$

Consider first Case (A) and let us write

$$\sum_{m=1}^{m=n} \frac{J_\nu(\lambda_m x)}{\lambda_m} = \sum_{m=1}^{m=p-1} + \sum_{m=p}^{m=n} = S_1 + S_2. \tag{99}$$

We have from (96)

$$|S_2| < \sum_{m=p}^{m=n} \frac{N}{\lambda_n^{\frac{3}{2}} x^{\frac{1}{2}}} < \frac{N}{\gamma x^{\frac{1}{2}}} \sum_{m=p}^{m=n} \frac{1}{\lambda_m^{\frac{3}{2}}} (\lambda_m - \lambda_{m-1}), \tag{100}$$

where γ is the lower limit of the distances between two successive positive roots of equation (2).* But since $\lambda_q x < k$ for values of x in the interval (98), and since p is the *greatest* integer such that $\lambda_p x < k$, it follows that $p \geqq q$, and we have from (95)

$$\sum_{m=p}^{m=n} \frac{1}{\lambda_m^{\frac{3}{2}}} (\lambda_m - \lambda_{m-1}) < \int_{\lambda_{p-1}}^{\lambda_n} \frac{dx}{x^{\frac{3}{2}}} < \frac{2}{\sqrt{\lambda_{p-1}}} = \frac{2}{\sqrt{\lambda_{p+1}}} \sqrt{\frac{\lambda_{p+1}}{\lambda_{p-1}}} < \frac{2^{\frac{3}{2}}}{\sqrt{\lambda_{p+1}}}. \tag{101}$$

Combining (100) and (101), we obtain, since $\lambda_{p+1} x \geqq k$,

$$|S_2| < \frac{N \cdot 2^{\frac{3}{2}}}{\gamma \sqrt{\lambda_{p+1}}\, x} \leqq \frac{2^{\frac{3}{2}} N}{\gamma \sqrt{k}} \qquad \left(0 < x < \frac{k}{\lambda_q}\right). \tag{102}$$

The discussion of S_1 is slightly different in the two cases $\nu > 1$ and $0 < \nu \leqq 1$. Consider first the case $\nu > 1$. We have

$$S_1 = \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x}\, x \leqq \frac{1}{\gamma} \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_{m+1} - \lambda_m) x. \tag{103}$$

But since, for $\nu > 1$, the function $J_\nu(y)/y$ is positive and steadily increases in the interval $0 < y < k$, where k has the same meaning as above, we have

$$\sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_{m+1} x - \lambda_m x) < \int_0^{\lambda_p x} \frac{J_\nu(y)}{y}\, dy < \int_0^k \frac{J_\nu(y)}{y}\, dy. \tag{104}$$

Combining (103) and (104), we obtain

$$0 < S_1 < \frac{1}{\gamma} \int_0^k \frac{J_\nu(y)}{y}\, dy \qquad \left(\nu > 1;\ 0 < x < \frac{k}{\lambda_q}\right). \tag{105}$$

* That such a lower limit exists and is different from zero is easily inferred from the fact that the limit of the distance between the nth and the $(n+1)$th roots of (2) exists and is equal to π.

Taking up next the case $0 < \nu \leqq 1$, we have

$$S_1 = \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} x < \frac{1}{\gamma_1} \sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_m - \lambda_{m-1}) x, \tag{106}$$

where we have set $\lambda_0 = 0$ for the sake of uniformity, and have used γ_1 to represent the smaller of the two quantities γ and λ_1. But since, for $0 < \nu \leqq 1$, the function $J_\nu(y)/y$ is positive and steadily decreases in the interval $0 < y < k$, where k has the same meaning as above, we have

$$\sum_{m=1}^{m=p-1} \frac{J_\nu(\lambda_m x)}{\lambda_m x} (\lambda_m x - \lambda_{m-1} x) < \int_0^{\lambda_{p-1} x} \frac{J_\nu(y)}{y} dy < \int_0^k \frac{J_\nu(y)}{y} dy, \tag{107}$$

and hence, combining (107) with (106),

$$0 < S_1 < \frac{1}{\gamma_1} \int_0^k \frac{J_\nu(y)}{y} dy \qquad \left(0 < \nu \leqq 1;\ 0 < x < \frac{k}{\lambda_q}\right). \tag{108}$$

From (105) and (108) we have

$$0 < S_1 < K \qquad \left(\nu > 0;\ 0 < x < \frac{k}{\lambda_q}\right), \tag{109}$$

where K is a positive constant. Combining (99), (102), and (109), we obtain

$$\left| \sum_{m=1}^{m=n} \frac{J_\nu(\lambda_m x)}{\lambda_m} \right| < K + \frac{2^{\frac{1}{2}} N}{\gamma \sqrt{k}} \qquad \left(0 < x < \frac{k}{\lambda_q}\right), \tag{110}$$

and hence in Case (A) the sum of the first n terms of the series whose general term is (91) remains finite for all values of n and all values of x in the interval (98). The treatment of Case (B) is entirely analogous to the treatment of S_1 in Case (A), and consequently we may infer that the sum of the first n terms of the series remains finite for all values of n and all values of x in the interval (98). Hence, as was pointed out before, our lemma is proved.

THEOREM I. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series* (6) *will, provided $\nu = 0$ or $f(0) = 0$, converge uniformly to $f(x)$ throughout any interval*

$$0 \leqq x \leqq x_0 < c_1, \tag{111}$$

where c_1 is the smallest value of x for which $f(x)$ has a discontinuity; provided the l of equation (7) *is not zero or $f(1)$ is zero, it will converge uniformly to $f(x)$ throughout any interval*

$$c_k < x_1 \leqq x \leqq 1, \tag{112}$$

where c_k is the largest value of x for which $f(x)$ has a discontinuity; finally, the sum of n terms of the series will remain finite for all values of n and all values of x lying in the interval

$$0 \leqq x \leqq 1. \tag{113}$$

In view of Lemma 1, the general term of the series (1), and hence the general term of the series (6),* may be written in the form

$$\sum_{m=1}^{m=k} M_m \frac{\cos\left[(n\pi+q)c_m - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} J_\nu(\lambda_n x)$$

$$+ Mf(1)\frac{\cos\left[(n\pi+q) - \frac{2\nu+3}{4}\pi\right]}{\sqrt{n}} J_\nu(\lambda_n x) \tag{114}$$

$$+ \frac{\nu K f(0)}{\lambda_n} J_\nu(\lambda_n x) + r_n J_\nu(\lambda_n x),$$

where the M's and K are constants, the c's are points of discontinuity of $f(x)$, and r_n is the general term of an absolutely convergent series.

If either $\nu = 0$ or $f(0) = 0$, the third term in (114) drops out, and it only remains to consider the other three terms. Since the Bessel function of any order remains finite for all values of the argument, it is obvious that the last term in (114) is the general term of a series that is uniformly convergent in the interval (111). It follows from Lemma 2 that the first and second terms of (114) are also the general terms of series that are uniformly convergent in this interval. Hence the whole expression (114) is the general term of such a series, and since (114) is only another way of writing the general term of (6), it follows that (6) is uniformly convergent in the interval (111). Since this is the case and since the terms of (6) are continuous in this interval, it must define there a continuous function of x, $\chi(x)$. However, for values of x lying in the interval $0 < x \leqq x_0$, (6) converges to $f(x)$,† i. e.,

$$\chi(x) = f(x) \qquad (0 < x \leqq x_0).$$

Both sides of the last equation have a limit as x approaches zero. Equating these limits, we get

$$\chi(0) = f(0).$$

Hence (1) converges uniformly to $f(x)$ throughout the interval (111), provided $\nu = 0$ or $f(0) = 0$, and the first part of our theorem is proved.

We wish next to show that, provided the l of equation (7) is not zero, or $f(1)$ is zero, the series (6) will converge uniformly to $f(x)$ throughout the interval (112).

Consider first the case $l \neq 0$. We saw in the previous discussion that the general term of (6) could be written in the form (114). It is obvious that the

* In the special case where (4) holds, (114) is the $(n+1)$th term of (6), but that of course makes no difference in a discussion of convergence.

† This, as we have stated earlier in this paper, has been established in previous discussions of the series (6). Cf. the articles by DINI, KNESER, and HOBSON quoted above.

third and fourth terms of (114) are the general terms of series that are uniformly convergent in the interval (112); it follows from Lemma 5 that the first term of (114) and, since the l of equation (7) is not zero, the second term of (114) are the general terms of series that are uniformly convergent in this interval. Hence the whole expression (114) is the general term of such a series, and since (114) is only another way of writing the general term of (6), it follows that (6) is uniformly convergent in the interval (112), provided the l of equation (7) is not zero.

Considering next the case of $f(1)=0$, we see that the second term of (114) is zero for this case, and is therefore the general term of a series that is uniformly convergent in the interval (112), whether the l of equation (7) is or is not zero. Hence, in this case also, (6) is uniformly convergent in the interval (112).

That the series (6) has the value $f(x)$ in this interval, under the conditions prescribed, was proved by HOBSON in the paper already quoted.

We come finally to the proof of the third part of our theorem, namely that the sum of n terms of the series (6) remains finite for all values of n and all values of x lying in the interval (113). Since r_n is the general term of an absolutely convergent series, the fourth term of (114) is obviously the general term of a series that is absolutely and uniformly convergent in the interval (113), and for which therefore the sum of n terms remains finite for all values of n and all values of x lying in this interval. It follows at once from Lemma 4 that the first and second terms of (114) are also the general terms of such series, and from Lemma 6 that the third term is the general term of such a series. Hence the whole expression (114), or its equivalent the general term of (6), is the general term of such a series.

Our theorem is, therefore, completely established.

The preceding theorem establishes the uniform convergence of the series (6) to $f(x)$ throughout the interval (111) only when $\nu=0$ or $f(0)=0$. That (6) could not possibly converge uniformly to $f(x)$ in the interval (111) in case neither of these restrictions was fulfilled is easy to see, since when $\nu>0$ the value of (6) for $x=0$ is always zero. *However, in the case* $\nu>0$, $f(0)\neq 0$, *it is possible to obtain a development of* $f(x)$ *that is uniformly convergent in the interval* (111), *in the form of a constant term plus a series of the type* (6). To do this, we have only to develop the function $\phi(x)=f(x)-f(0)$ in a series of the form (6), and precede this development by the constant term $f(0)$.

The reasoning in Theorem I is also *immediately* applicable to proving that *under the given conditions on* $f(x)$ *the series* (6) *will converge uniformly in any closed sub-interval of* $0<x<1$ *that does not include a point of discontinuity of* $f(x)$. This uniform convergence has been previously established,*

* By KNESER for developments in terms of J_0; by HOBSON for developments in terms of $J_\nu(\nu\geqq 0)$. Cf. the references given above.

but it seems worth while to point out the fact that it can be obtained independently from the results of the present paper.

THEOREM II. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series*

$$A_1 e^{-\lambda_1 z} F_0(\lambda_1, x) + A_2 e^{-\lambda_2 z} F_0(\lambda_2, x) + \cdots, \tag{115}$$

where the A's are the coefficients of the series (6) *for the case $\nu = 0$, and the λ's are the roots, positive or zero, of equation* (7) *arranged in increasing order of magnitude, will converge absolutely in the region*

$$\begin{pmatrix} 0 \leqq x \leqq 1 \\ z > 0 \end{pmatrix}; \tag{116}$$

its value $u(z, x)$ will be continuous there, will approach $f(x)$ as we approach any point of the interval $0 \leqq x < 1$ lying on the line $z = 0$, at which $f(x)$ is continuous, will approach $f(1)$ as we approach the point $x = 1$, $z = 0$, if the l of equation (7) *is not zero and $f(x)$ is continuous at $x = 1$, and finally will remain finite when we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.*

We obtain at once from the definition of A_n and Lemma 1

$$|A_n| = |A_n'| < K_1 \qquad (n = 1, 2, \cdots),$$

where K_1 is a positive constant. Then since $F_0(\lambda, x)$ remains finite for all values of x and λ, we have for the general term of (115), if we make use of (18),

$$|A_n e^{-\lambda_n z} F_0(\lambda_n, x)| < \frac{K_1 |F_0(\lambda_n, x)|}{e^{\lambda_n z}} < \frac{K}{e^{\lambda_n z_0}} < \frac{K}{(\lambda_n z_0)^2} < \frac{K}{C^2 z_0^2 n^2} \quad (z \geqq z_0 > 0),$$

where K is a positive constant. Hence (115) converges absolutely and uniformly for all values of x and z lying in the region

$$\begin{pmatrix} 0 \leqq x \leqq 1 \\ z \geqq z_0 > 0 \end{pmatrix}. \tag{117}$$

Since also its terms are continuous in this region, it represents a function that is continuous there or, since z_0 may be taken as any constant > 0, is continuous in the region (116).

At any point of the interval $0 \leqq x < 1$ at which $f(x)$ is continuous, and at the point $x = 1$ when the l of equation (7) is not zero and $f(x)$ is continuous there, the series (6) converges to $f(x)$. Hence, for values of x in the interval $0 \leqq x < 1$ at which $f(x)$ is continuous, and for $x = 1$ in the case $l \neq 0$ and $f(x)$ is continuous at $x = 1$, $u(z, x)$ will approach $f(x)$ as z approaches $+0$, since the convergence factors

$$e^{-\lambda_1 z}, \ e^{-\lambda_2 z}, \ \cdots$$

satisfy the conditions of a theorem due to Bromwich.* Furthermore, (6) converges uniformly to $f(x)$ throughout any closed interval, in which $f(x)$ is continuous, that lies in the interval $0 \leqq x \leqq 1$ and does not include the point $x = 1$ for the case where the l of equation (7) is zero. Consequently $u(z, x)$ will approach $f(x)$ *uniformly* for all values of x lying in such an interval as z approaches $+0$, as can be shown by a slight modification of the reasoning in Bromwich's theorem. Hence $u(z, x)$ will approach $f(x)$ as we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$ at which $f(x)$ is continuous, † along any path lying in the region (116).

It remains finally to be shown that $u(z, x)$ remains finite as we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.

If we represent by $s_n(x)$ the sum of the first n terms of the series (6), we may write the series (115) in the form

$$\sum_{n=1}^{\infty} [s_n(x) - s_{n-1}(x)]\, e^{-\lambda_n z}, \tag{118}$$

where for the sake of uniformity we have taken $s_0(x) = 0$. We know from Theorem I that $s_n(x)$ remains finite for all values of n and all values of x lying in the interval (113). Consequently, since for $z > 0$

$$\lim_{n=\infty} e^{-\lambda_n z} = 0,$$

the series obtained by dropping the parentheses in the series (118) is one for which the limit of the general term is zero. Hence, since no parenthesis contains more than a fixed number of terms, we may drop the parentheses without changing the value of the series. Rearranging the terms in such a manner that

* Mathematische Annalen, vol. 65 (1908), p. 358. BROMWICH's conditions are

$$\left.\begin{array}{l} \sum_{n=\mu}^{n=\nu} n^k |\Delta^{k+1} v_n| < K \\ \lim_{n=\infty} n^k v_n = 0 \end{array}\right\} a > 0,$$

$$\lim_{a=+0} v_n = 1,$$

where μ and ν are two integers, $v_1, v_2, v_3, \cdots$ are the convergence factors, $\Delta^{k+1} v_n$ represents their $(k+1)$th differences, and K is a positive constant. When the convergence factors are introduced into a convergent series, as in the present instance, the k of this theorem must be taken equal to zero.

A simple proof of the special case here dealt with of BROMWICH's theorem can be obtained from the expression (119), which is equal to the series (115), by throwing it into the form

$$f(x)\, e^{-\lambda_1 z} + \sum_{n=1}^{n=\infty} \varepsilon_n(x) [e^{-\lambda_n z} - e^{-\lambda_{n+1} z}],$$

where

$$\varepsilon_n(x) = s_n(x) - f(x).$$

† We must except, of course, the point $x = 1$ for the case where the l of equation (7) is equal to zero.

no term is carried over more than a fixed number of terms and reinserting parentheses does not change the value of the series, and hence (118) may be written in the form

$$\sum_{n=1}^{\infty} s_n(x)\left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right]. \tag{119}$$

But since

$$|s_n(x)| < M \qquad (n=1, 2, \cdots; 0 \leqq x \leqq 1),$$

where M is a positive constant, we have, in view of the fact that (119) converges to $u(z, x)$,

$$|u(z, x)| \leqq \sum_{n=1}^{\infty} |s_n(x)| \cdot \left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right] < M \sum_{n=1}^{\infty} \left[e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right] = Me^{-\lambda_1 z},$$

and consequently $u(z, x)$ remains finite when we approach any point of the interval $0 \leqq x \leqq 1$ lying on the line $z = 0$.

Our theorem is, therefore, completely established.

A single example will show clearly the manner in which the above results can be applied to problems in mathematical physics. Let us consider the problem of determining the steady flow of heat in a semi-infinite circular cylinder whose base is kept at a temperature

$$v = f(r),$$

where r is the distance from the axis of the cylinder.

This problem reduces to the determination of a function $v(r, z)$ which is finite throughout the region $z \geqq 0$, $0 \leqq r \leqq 1$, and is continuous throughout this region except when $z = 0$ and r has a value for which $f(r)$ is discontinuous; which satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} = 0 \tag{120}$$

in the region

$$\begin{pmatrix} z > 0 \\ 0 < r < 1 \end{pmatrix}; \tag{121}$$

and which also satisfies the conditions

$$l\frac{\partial v}{\partial r} + hv = 0 \qquad (z > 0, r = 1) \tag{122}$$

and

$$v = f(r) \qquad (z = 0).^* \tag{123}$$

It can be shown easily that the function

$$Ae^{-\lambda z}F_0(\lambda, r), \tag{124}$$

where A is a constant and λ is a root of equation (7), satisfies (120) and (122).

* Cf. CARSLAW, *Introduction to the Theory of Fourier's Series and Integrals, etc.*, Chap. XV, p. 321.

It remains to construct from functions of the form (124) a continuous function which will also satisfy (120) and (122) and in addition will approach the boundary value $f(r)$, where $f(r)$ is continuous, and which will remain finite where that is not the case.

Consider the series

$$A_1 e^{-\lambda_1 z} F_0(\lambda_1, r) + A_2 e^{-\lambda_2 z} F_0(\lambda_2, r) + \cdots, \tag{125}$$

where the λ's are the roots, positive or zero, of equation (7), arranged in increasing order of magnitude, and the A's are the coefficients of the series (6) for $\nu = 0$ corresponding to the given function $f(r)$. It follows from Theorem II that if $f(r)$ satisfies the conditions imposed on $f(x)$ in Lemma 1, this series will converge in the region

$$\begin{pmatrix} z > 0 \\ 0 \leqq r \leqq 1 \end{pmatrix} \tag{126}$$

and define there a continuous function $u(r, z)$; that this function satisfies (120) and (122) can be shown in a manner similar to that in which its convergence and continuity were established. Finally it follows from Theorem II that $u(r, z)$ will approach $f(r)$ when we approach a point on the line $z = 0$ at which $f(r)$ is continuous,* and that it will remain finite when we approach any other point of this line. Thus $u(r, z)$ forms a complete solution of the physical problem.†

THE UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.

* We must except here the point $r = 1$ for the case in which the l of equation (7) is zero. The necessity for this exception is now apparent from the physical point of view as well as the mathematical one, for the case $l = 0$ corresponds to the case in which the convex surface of the cylinder is kept at temperature zero, and hence we could not approach the temperature at the circumference of the base of the cylinder as we approached this circumference along *any* path except in the very special case that this temperature was zero.

† In the discussion of other problems of mathematical physics that involve developments in Bessel functions of order zero, we may encounter convergence factors slightly different from those of Theorem II. However, in most cases the necessary modifications in Theorem II will be only those of form, since the convergence factors will satisfy the same general theorem that was used in the proof of Theorem II.

DETERMINATION OF THE ORDINARY AND MODULAR TERNARY LINEAR GROUPS*

BY

HOWARD H. MITCHELL

INTRODUCTION.

It is proposed in this paper to determine the finite groups of collineations in three homogeneous variables, a collineation being of the form

$$\rho x_i' = \sum_{j=1}^{3} a_{ij} x_j \qquad (i = 1, 2, 3).$$

The determination will be made both in the case where the coefficients a_{ij} are ordinary numbers and in the case where they are marks of a finite Galois field, $GF(p^n)$, p being an odd prime and n a positive integer. The variables x_i are therefore regarded as homogeneous coördinates of the points either of the ordinary plane or of a modular plane,† $PG(2, p^n)$.

The finite groups of collineations in the ordinary plane have been already determined.‡ The treatment of the problem in this paper is however different from any thus far given. It is based almost entirely on geometrical methods and it is hoped may prove of interest. The discussion for this case is contained in §§ 2–10.

Considerable work has been done on the ternary modular groups. For the case where the coefficients of the transformations are in the $GF(p)$, the question was considered by BURNSIDE.§ For the same case a complete determination of those groups which contain operators of period p was made by DICKSON.‖

* Presented to the Society, under different titles, December 30, 1909, April 30 and September 7, 1910.

† O. VEBLEN and W. H. BUSSEY, *Finite Projective Geometries*, these Transactions, vol. 7 (1906), pp. 241–259.

‡ For a bibliography of this subject consult WIMAN, *Endliche Gruppen linearer Substitutionen, Encykopädie der Mathematischen Wissenschaften*, Bd. I, pp. 528–530. The first accurate solution of the problem is due to BLICHFELDT, *On the Order of Linear Homogeneous Groups*, these Transactions, vol. 4 (1903), pp. 387–397, and vol. 5 (1904), pp. 310–325; also *The Finite, Discontinuous, Primitive Groups of Collineations in Three Variables*, Mathematische Annalen, vol. 63 (1907), pp. 552–572.

§ Proceedings of the London Mathematical Society, vol. 26 (1895), pp. 58–106.

‖ American Journal of Mathematics, vol. 27 (1905), pp. 189–202.

For the case where the coefficients are in the $GF(p^2)$, a discussion of those groups whose orders are divisible by p^6, p^5 and p^4 was given by R. L. BÖRGER.*

As a result of the determination of the ternary modular groups in this paper, the subgroups of the two systems of simple groups, $LF(3, p^k)$ and $HO(3, p^{2k})$, are found in the cases where p is an odd prime. † The $LF(3, p^k)$ is the group of all ternary transformations the coefficients of which lie in the $GF(p^k)$ and which have for determinant a cube in that field. It is identical with the group of all collineations of the modular plane, $PG(2, p^k)$, if $p^k - 1$ is not divisible by 3, and is a self-conjugate subgroup of that group of index 3 if $p^k - 1$ is divisible by 3. Its order is

$$\frac{1}{\mu}(p^{2k} + p^k + 1)(p^k + 1)p^{3k}(p^k - 1)^2,$$

where μ is the greatest common divisor of 3 and $p^k - 1$. The $HO(3, p^{2k})$ is a subgroup of the $LF(3, p^{2k})$ having an invariant of the form

$$x_1^{p^k+1} + x_2^{p^k+1} + x_3^{p^k+1} = 0.$$

Its order is

$$\frac{1}{\nu}(p^{2k} - p^k + 1)(p^k + 1)^2 p^{3k}(p^k - 1),$$

where ν is the greatest common divisor of 3 and $p^k + 1$.

The binary modular groups have been fully determined. For the case where the coefficients are in the $GF(p)$ the determination was made by GIERSTER.‡ In the general case the problem has been solved by E. H. MOORE§ and by WIMAN.‖ A treatment based on these two papers is given by DICKSON (*Linear Groups*, Chap. XII). A new treatment will be given in this paper.

The writer wishes to acknowledge his indebtedness to Professor O. VEBLEN of Princeton University for constant and valuable aid in the preparation of the paper.

§1. BINARY GROUPS.

We consider transformations in two homogeneous variables of the form

$$\rho x_i' = \sum_{j=1}^{2} a_{ij} x_j \qquad (i = 1, 2).$$

* *Ibid.*, vol. 32 (1910), pp. 289–298.

† See DICKSON, *Linear Groups*, pp. 75–78, 126–144.

‡ Mathematische Annalen, vol. 18 (1881).

§ *The Subgroups of the Generalized Finite Modular Group, Decennial Publications of the University of Chicago*, vol. 9 (1904), pp. 141–190.

‖ *Bestimmung aller Untergruppen einer doppelt unendlichen Reihe von einfachen Gruppen*, Bihang till K. Svenska Vet.-Akad. Handlingar, vol. 25, part 1, no. 2.

These transformations may be regarded as permuting the points $(x_1 x_2)$ of a line (either ordinary or modular). Any such transformation leaves invariant either one or two points on the line. In the case of the ordinary line any transformation of finite period leaves invariant two points. The groups containing such transformations only are well known. They are cyclic groups of order d, dihedral groups of order $2d$, tetrahedral groups of order 12, octahedral groups of order 24, and icosahedral groups of order 60. No further discussion will be given of these groups.

We therefore consider those groups on the modular line which contain transformations leaving invariant a single point. Any such transformation is additive and of period p.* Any group containing these additive transformations will contain at least one additive group, which leaves invariant a point and contains all the additive transformations which have that point for fixed point. The order of any such additive group is a power of p, say p^m. If the fixed point be (10), the additive group is represented by $[x_1 + \lambda x_2,\ x_2]$, where λ takes all the values in an additive field $\alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_m\lambda_m$, the α's being integers $0, 1, 2, \cdots, p-1$, and each λ_j a mark not included in the additive field $\alpha_1\lambda_1 + \cdots + \alpha_{j-1}\lambda_{j-1}$. There will be conjugate with it $1 + fp^m$ additive groups, f being 0 or a positive integer. All additive transformations must therefore lie in these conjugate additive groups.

The additive group will be self-conjugate under a maximum metacyclic group of order $d_1 p^m$, where d_1 is a factor of $p^m - 1$ (in special cases we may have $d_1 = 1$). Such a metacyclic group contains p^m conjugate cyclic groups, each of which leaves invariant two points and is of period d_1. If (01) be the other fixed point of one of these cyclic groups, the metacyclic group may be generated by the additive group together with $[\eta x_1,\ x_2]$, where η is of period d_1 and belongs to the multiplier field of the additive field. If Ω denotes the order of the whole group there will be $\Omega/d_1 p^m$ additive groups containing $(p^m - 1)\Omega/d_1 p^m$ additive transformations. There will also be in the group Ω/d_1 or $\Omega/2d_1$ cyclic groups of order d_1 containing $(d_1 - 1)\Omega/d_1$ or $(d_1 - 1)\Omega/2d_1$ transformations, according as one of those cyclic groups is self-conjugate under itself only or under a dihedral group interchanging its two fixed points. Any other maximal cyclic group of order d_i, which does not lie in a metacyclic group, will be self-conjugate either under itself only or under a dihedral group of order $2d_i$ interchanging its two fixed points. Conjugate with such a cyclic group there will be Ω/d_i or $\Omega/2d_i$ cyclic groups containing $(d_i - 1)\Omega/d_i$ or $(d_i - 1)\Omega/2d_i$ transformations.

The order of the group will be equal to the number of transformations which it contains. Hence Ω must satisfy a Diophantine equation of the following

*Throughout the paper p denotes the modulus. The discussion of the binary groups applies also to the case $p = 2$.

form:*

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^m} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \qquad (f_i = 1, 2).$$

The coefficient of Ω on the right of the above equation must be less than unity. Also, since the group contains a subgroup of order $d_1 p^m$, the coefficient of Ω on the right must be equal to or greater than $(d_1 p^m - 1)/d_1 p^m$. If $d_1 = 1$, we find therefore that either the final sum is absent, or it contains but one term, in which case $f_2 = 2$. If $d_1 > 1$, either $r = 1$, $f_1 = 1$, or $r = 2$, $f_1 = f_2 = 2$.

If $d_1 = 1$ and the final sum is absent, we obtain $\Omega = p^m$, i. e., a single additive group.

If $d_1 = 1$ and there is one term in the final sum, we have either $p^m = 2$, $\Omega = 2d_2$, which represents a dihedral group, or $p^m = 3$, $d_2 = 2$, $\Omega = 12$, which represents a tetrahedral group.

If $d_1 > 1$, $r = 1$, $f_1 = 1$, we obtain $\Omega = d_1 p^m$, which represents a single metacyclic group.

If $d_1 > 1$, $r = 2$, $f_1 = f_2 = 2$, we shall show that d_1 and d_2 cannot contain any common factor except 2. The cyclic groups of order d_1, when transformed by one of their number, will be permuted in cycles of period d_1 or $d_1/2$. Hence the total number of conjugate groups of order d_1 must be of the form $1 + fd_1/2$. If we suppose the groups of order d_1 to be transformed by one of the groups of order d_2, it is evident that the number of the former must also be of the form $f'd_2/2$. Hence d_1 and d_2 can contain no common factor except 2.

The order, Ω, must be the least common multiple of p^m, $2d_1$, and $2d_2$. For it must be divisible by the least common multiple, since the group contains subgroups of those orders. Moreover if we collect the three terms on the right and write the equation

$$\Omega = 1 + \frac{I}{M}\Omega,$$

where M denotes the least common multiple and I is an integer, it is clear that Ω cannot be greater than M.

Hence, if p is odd, Ω must have one of the two values, $d_1 d_2 p^m$, and $2d_1 d_2 p^m$, according as d_1 and d_2 are both divisible by 2 or have no common factor. For $p = 2$, $\Omega = d_1 d_2 p^m$.

Putting $\Omega = d_1 d_2 p^m$ in the equation, we obtain

$$(d_2 - d_1)p^m - 2(d_2 - 1) = 0.$$

Hence

$$d_2 = fp^m + 1, \qquad d_1 = f(p^m - 2) + 1,$$

* This type of Diophantine equation originated in C. JORDAN's attempt to determine all finite collineation groups in the ordinary plane, Journal für die reine und angewandte Mathematik, vol. 84 (1878), p. 89.

where f denotes an integer. But d_1 is a divisor of p^m-1. Hence

$$f=1,\ d_1=p^m-1,\ d_2=p^m+1,\ \Omega=(p^m+1)p^m(p^m-1).$$

Putting $\Omega=2d_1d_2p^m$ in the equation, we obtain

$$(d_2-d_1)p^m-(2d_2-1)=0.$$

Hence

$$d_2=\tfrac{1}{2}(fp^m+1),\qquad d_1=\tfrac{1}{2}(fp^m+1)-f,$$

where f denotes an odd integer. But d_1 is a divisor of p^m-1. Hence

$$f=1,\ d_1=\frac{p^m-1}{2},\ d_2=\frac{p^m+1}{2},\ \Omega=\tfrac{1}{2}(p^m+1)p^m(p^m-1);$$

or $f=3,\ p^m=3,\ d_1=2,\ d_2=5,\ \Omega=60$.

We consider the possible group of order $(p^m+1)p^m(p^m-1)$. Each additive group of order p^m is self-conjugate under a metacyclic group of order $(p^m-1)p^m$. Hence there are p^m+1 such additive groups. The p^m+1 fixed points of these additive groups must therefore be permuted among themselves. The two fixed points of each cyclic group of order p^m-1 are interchanged by transformations of the group, i. e., they are among the p^m+1 points. The additive group leaving fixed one of these points is transitive on the rest.

Since three points may be sent by a transformation into any other three, we may choose three of the p^m+1 points as $(01), (10), (11)$. The following three transformations will then be in the group: $[\eta x_1, x_2]$, $[x_1+x_2, x_2]$, $[x_1, x_1+x_2]$, where the period of η is p^m-1. The powers of η together with 0 form the Galois field, $GF(p^m)$. The three transformations generate a group of order $(p^m+1)p^m(p^m-1)$, which is the group of all transformations with coefficients in the $GF(p^m)$.

We may construct the group of order

$$\tfrac{1}{2}(p^m+1)p^m(p^m-1)$$

by replacing η by η^2. It is the group of all transformations with coefficients in the $GF(p^m)$ and having for determinant a square in that field.

The group of order 60 is an icosahedral group. For it contains $\Omega/2\cdot 2=15$ involutions, each of which lies in one four-group. Hence the fifteen involutions arrange themselves in five four-groups. Each involution leaves fixed the four-group in which it lies and permutes the other four in pairs. The group must then be a G_{60}^5, i. e., an icosahedral group. Such a group may be generated by the three operators, E_1: $[x_1+x_2, x_2]$, E_2: $[ix_1, -x_1-ix_2]$, E_3: $[-ix_1, ix_2]$ $(i^2=-1)$, which satisfy the generational relations:*

$$E_1^3=E_2^2=E_3^2=(E_1E_2)^3=(E_1E_3)^2=(E_2E_3)^3=I.$$

*E. H. Moore, Proceedings of the London Mathematical Society, vol. 27 (1897), pp. 357-366; Dickson, *Linear Groups*, p. 289.

§ 2. Canonical Forms of Transformations in the Plane.*

The transformations in the plane† may be classified according to the five types of invariant figures. Those of type I leave fixed a triangle; those of type II leave fixed two points and two lines; those of type III leave fixed a lineal element; those of type IV leave fixed all the points of a line and all the lines through a point off that line; those of type V leave fixed all the points of a line and all the lines through a point on that line (Fig. 1).

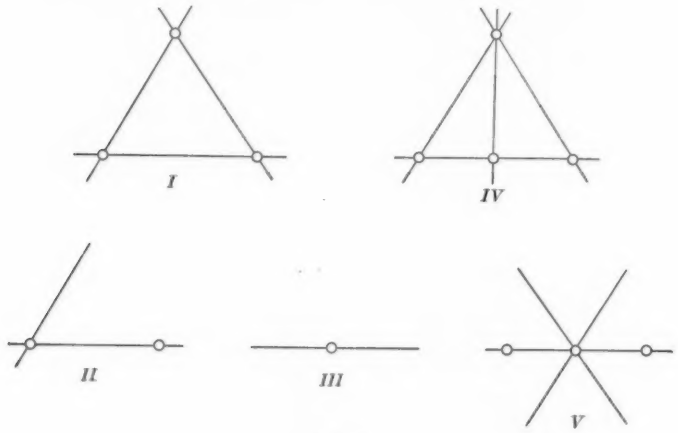

Fig. 1.

A transformation of type I may be written in canonical form $[\alpha x_1, \beta x_2, \gamma x_3]$. This transformation is of finite period in the ordinary plane provided the ratios of the three quantities, α, β, γ, are roots of unity (not unity itself).

A transformation of type II may be written $[\alpha x_1, x_2 + x_3, x_3]$, where $\alpha \neq 0, 1$. The period of this transformation is infinite in the ordinary plane and in the modular plane contains p as a factor.

A transformation of type III may be written $[x_1 + x_2, x_2 + x_3, x_3]$. The period is infinite in the ordinary plane and equal to p in the modular plane (since $p > 2$).

A transformation of type IV may be written $[\alpha x_1, x_2, x_3]$, where $\alpha \neq 0, 1$. In the ordinary plane it is of finite period if α is a root of unity. We will refer to a transformation of this type as an *homology*. In particular, if the period is 2, i. e., if $\alpha = -1$, we will refer to it as a *reflection*.

A transformation of type V may be written $[x_1, x_2 + x_3, x_3]$. In the ordinary plane it is of infinite period and in the modular plane of period p. We

* Cf. Veblen and Young, *Projective Geometry*; Dickson, *Linear Groups*, chaps. X, XI.

† For the modular plane the discussion will be limited to the case where p is an odd prime.

shall refer to a transformation of this type as an *elation*. Any power d of the transformation of type II such that $\alpha^d = 1$ is an elation.

§ 3. General Theorems.

Theorem 1. *A group leaving fixed a line and represented on the points of that line by a four-group contains a reflection with that line as axis; dually, a group leaving fixed a point and represented on the lines through the point by a four-group contains a reflection with that point as center.*

We shall prove the first half of this theorem. We choose the fixed line as $x_3 = 0$ and the fixed pair of points of one involution of the four-group as (100) and (010). If then we choose as (110) one of the fixed points of another of the three involutions, the other fixed point will be (1–10). Then any two transformations which are represented on the fixed line by these two involutions are of the form

$$[\alpha x_1 + \gamma_1 x_3,\ -\alpha x_2 + \gamma_2 x_3,\ x_3], \qquad [\beta x_2 + \delta_1 x_3,\ \beta x_1 + \delta_2 x_3,\ x_3].$$

Their product is $[\alpha\beta x_2 + \epsilon_1 x_3,\ -\alpha\beta x_1 + \epsilon_2 x_3,\ x_3]$. This is a transformation which is represented on the fixed line by the third involution of the four-group.

The square of any one of these three transformations will leave fixed all the points on $x_3 = 0$ and hence will be either an homology or an elation with that line as axis or else the identity. One at least of them, however, must be an homology of even period, since if the periods of α^2 and β^2 are both odd the period of $-\alpha^2\beta^2$ will be even. An homology of even period will contain as a power a reflection.

The dual theorem may be proved in a similar manner.

Theorem 2. *A group permuting cyclically the vertices of a triangle and represented on each side of the triangle by a cyclic group of order d is of order $3dd'$, where d' is a factor of d which is divisible by all the prime factors of d of the form $3f-1$ and all the factors 3 with the exception of at most one.*

We choose the fixed triangle as the triangle of reference. A transformation which is of period d on $x_3 = 0$ may be written $[x_1, \omega x_2, \omega^e x_3]$, where ω is of period d, the coefficient of x_3 being a power of ω since the period of the transformation on $x_1 = 0$ and $x_2 = 0$ must be a factor of d. If we transform this transformation by a transformation permuting cyclically the vertices of the fixed triangle, we obtain $[x_1, \omega^{-e} x_2, \omega^{1-e} x_3]$. The eth power of the first transformation by the second is the homology $[x_1, x_2, \omega^{e^2-e+1} x_3]$.

From the theory of quadratic forms it follows that $e^2 - e + 1$ cannot be divisible by any primes of the form $3f-1$ or by 3 to a higher power than the first. The period of the homology is then divisible by all the prime factors of the form $3f-1$ and all the factors 3 with the exception of at most one.

Theorem 3. *A group making all six permutations on the vertices of a tri-*

angle and represented on each side of the triangle by a dihedral group of order $2d$ *is of order* $2d^2$ *or* $6d^2$ *if* d *is divisible by* 3 *and of order* $6d^2$ *if* d *is not divisible by* 3.

We choose the fixed triangle as the triangle of reference. The group will contain a transformation of period d on $x_3 = 0$ of the form, $[x_1, \omega x_2, \omega^e x_3]$, where ω is of period d. A transformation interchanging $x_1 = 0$ and $x_2 = 0$ and leaving $x_3 = 0$ fixed transforms this into $[x_1, \omega^{-1} x_2, \omega^{e-1} x_3]$. As a product we obtain the homology (or the identity), $[x_1, x_2, \omega^{2e-1} x_3]$. Similarly there is in the group the homology, $[x_1, x_2, \omega^{2-e} x_3]$. The product of the first homology by the square of the second gives $[x_1, x_2, \omega^3 x_3]$. This homology is of period $d/3$ or d according as d is or is not divisible by 3.

In particular if there are no homologies with centers at the vertices of the triangle and having for axes the opposite sides, we must have $d = 3$. If there are only reflections, we must have $d = 2, 6$.

§ 4. Multiplicative Groups containing only Homologies and Transformations of Type I; General Properties.

Since homologies and transformations of type I are called multiplicative, groups containing transformations of those types only will be referred to as *multiplicative* groups. Any group in the ordinary plane is thus multiplicative.

Theorem 4. *In a multiplicative group a transformation which leaves fixed the center of an homology must leave fixed its axis and vice-versa.*

The product of an homology and the transformed of its inverse by a transformation leaving fixed its center but not its axis is an elation.

Theorem 5. *No multiplicative group can contain two homologies such that the line joining their centers passes through the intersection of their axes.*

The product of two homologies such that the line joining the centers passes through the intersection of their axes is of type II or III.

§ 5. Multiplicative Groups containing Homologies of higher Period than 3.

Theorem 6. *No multiplicative group which does not leave invariant a point, line, or triangle can contain homologies of period greater than* 5.

A group which does not leave invariant a point, line, or triangle and which contains homologies of period greater than 5 will contain two such homologies which are not commutative. These two homologies will leave invariant the point of intersection of their axes and the line joining their centers. But they cannot generate on the line joining their centers a group which is cyclic, dihedral, tetrahedral, octahedral, or icosahedral. Hence no group can contain homologies of period greater than 5.

Theorem 7. *No multiplicative group which does not leave invariant a point, line, or triangle can contain homologies of period* 5.

Consider two homologies of period 5 which are not commutative. They must generate the icosahedral group on the line joining their centers. Since there are in the icosahedral group involutions interchanging the fixed points of the C_5 there will be two homologies of period 5 represented on the fixed line by the same C_5. As a product we may then obtain an homology of period 5 having the fixed line for axis. But there will also be a reflection having that line for axis, since the icosahedral group contains a four-group as a subgroup (Theorem 1). There must then be an homology of period 10 having that line for axis, which is impossible (Theorem 6).

Theorem 8. *No multiplicative group which does not leave fixed a point, line, or triangle can contain homologies of period* 4.

Consider two homologies of period 4 which are not commutative. They must generate the octahedral group on the line joining their centers. The centers and axes of the two homologies must then harmonically separate each other. We choose them as (010) and $x_2 = 0$, (01–1) and $x_2 - x_3 = 0$. Since in a G_{24} there are involutions interchanging the two fixed points of a C_4 there is an homology with center (001) and axis $x_3 = 0$ and consequently an homology with center (100) and axis $x_1 = 0$. Hence there must be an axis joining any two centers. Consequently, since but six axes can pass through (100), there can be no homologies the centers of which do not lie on one of those six axes. Since we assume no line remains invariant under the group there must be other centers on each of the six axes. If a center on $x_2 = 0$ be chosen as (10–1), the corresponding axis must harmonically separate (100) and $x_1 = 0$, and will therefore be $x_1 - x_3 = 0$. But (10–1) and $x_1 - x_3 = 0$ do not harmonically separate (01–1) and $x_2 - x_3 = 0$. The two homologies having them for centers and axes cannot then generate an octahedral group on $x_1 + x_2 + x_3 = 0$. No group containing homologies of period 4 is therefore possible.

§6. Multiplicative Groups containing Homologies of Period 3.

Theorem 9. *The Hessian group G_{216} is the only multiplicative group containing homologies of period* 3, *which does not leave invariant a point, line, or triangle.*

A group which contains homologies of period 3 and which does not leave invariant a point, line, or triangle, will contain two such homologies which are not commutative. We choose the center and axis of the first as (100) and $x_1 = 0$, and the center of the second as (111). If the point of intersection of the two axes be chosen as (01–1), the axis of the second will be some line of the pencil, $Ax_1 + x_2 + x_3 = 0$. The group of the points on the line, $x_2 - x_3 = 0$, which joins the two centers, must be the tetrahedral group G_{12}. For if it were the octahedral or icosahedral group there would be an homology of period 3 having $x_2 - x_3 = 0$ for axis, since in either of these groups there are involutions

interchanging the fixed points of the C_3. But since there must be a reflection having that line for axis (Theorem 1), there would then be an homology of period 6 having that line for axis, which is impossible (Theorem 6). In order that the two homologies shall generate the tetrahedral group on $x_2 - x_3 = 0$, the axis of the second must be $x_1 + x_2 + x_3 = 0$. The two homologies are then

$$[\omega x_1,\ x_2,\ x_3],$$

$$x_1' = (\omega + 2)x_1 + (\omega - 1)x_2 + (\omega - 1)x_3,$$

$$x_2' = (\omega - 1)x_1 + (\omega + 2)x_2 + (\omega - 1)x_3, \quad (\omega^2 + \omega + 1 = 0).$$

$$x_3' = (\omega - 1)x_1 + (\omega - 1)x_2 + (\omega + 2)x_3$$

There are then on $x_2 - x_3 = 0$ the four centers (100), (111), (ω11), ($\omega^2$11), and there are four axes $x_1 = 0$, $x_1 + x_2 + x_3 = 0$, $\omega^2 x_1 + x_2 + x_3 = 0$, $\omega x_1 + x_2 + x_3 = 0$, passing through (01–1). The group is of order 24, having (2, 1) isomorphism with the tetrahedral group. The transformations which are represented by C_2 on $x_2 - x_3 = 0$ are C_4 in the plane. Together these three C_4 form the quaternion G_8.

In any group containing this group of order 24 every other homology of period 3 must be commutative with one of the above four. For consider one which is not. It cannot generate with the reflection with center (01–1) and axis

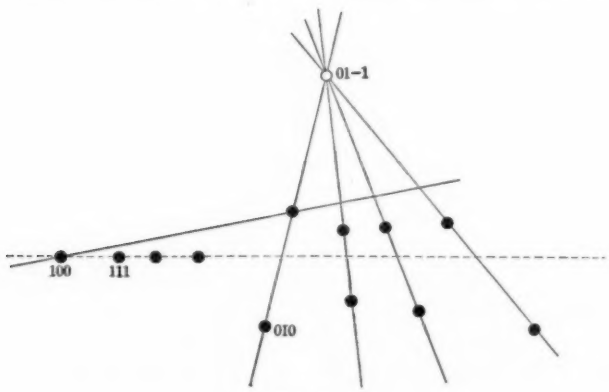

FIG. 2.

$x_2 - x_3 = 0$ a tetrahedral group on the line joining their centers, since we have seen that the transformations in the plane which are represented by C_2 in the tetrahedral group are C_4 in the plane. The group on the line joining the centers must then be dihedral. But this will involve an homology of period 3 having for axis the line joining the centers. But since the group of the lines through

(01–1) must be the tetrahedral group, there can be but four axes through it. There can be therefore no such homology.

There can be but two homologies of period 3 commutative with each of the four homologies in the group of order 24, since if the group of the points on one of the four axes is not dihedral, there will be a reflection having that line for axis (Theorem 1) and hence an homology of period 6 having that line for axis. If a center on $x_1 = 0$ be chosen as (010), the corresponding axis must pass through (100) and with the center harmonically separate (01–1) and $x_2 - x_3 = 0$, which are the center and axis of the reflection (Fig. 2). The homology is then $[x_1, \omega x_2, x_3]$.

There are then twelve homologies of period 3 in the group which form by threes four triangles. The group leaving one of the triangles point-wise invariant is of order 9, all six permutations are made on its vertices, and the group is transitive on the triangles. Its order is therefore $9 \cdot 6 \cdot 4 = 216$. It is the well-known Hessian group, permuting among themselves the nine inflexional points of a cubic curve. In this case they are (01–1), (01–ω), (01–ω^2), (10–1), (10–ω), (10–ω^2), (1–10), (1–ω0), (1–$\omega^2$0). It is the largest group which leaves invariant the Abelian G_9 generated by $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$. From the unique choice of coördinates it follows that there is a single conjugate set of G_{216} under the whole collineation group of the plane.

§ 7. Multiplicative Groups which contain Reflections; General Properties.

Having made an exhaustive study of multiplicative groups which contain homologies of higher period than 2, we will assume in the remaining discussion of those groups that the only homologies which are present are reflections. We suppose the existence of a group which contains reflections and consider a particular reflection in the group. We denote its center and axis by A and a respectively. There will be a group commutative with it which is (2, 1) isomorphic with the group of the points on a. There will be points conjugate with A under the group, some of which will not lie on a, since we assume that no triangle is left invariant. The points conjugate with A will lie on lines through A which will form one or more conjugate sets. Such a line may be an axis of a reflection, in which case its center will lie on a (Theorem 5). There will then be a center at its intersection with a, which may or may not be conjugate with A.

Consider a line through A which contains centers of reflections and which is not itself an axis of a reflection. There cannot then be a reflection with center at its intersection with a. The group of the points on this line cannot contain a four-group as a subgroup and hence will be dihedral, containing a cyclic base of odd order. Since a dihedral group may be generated by two reflections, the axes of all the reflections, whose centers lie on the line, will pass through a point.

All the reflections with centers on the line will be conjugate. We shall refer to a line of this character as an o-line.

There will be a cyclic group of odd order, d, having the o-line for one of the sides of its fixed triangle. This cyclic group will be self-conjugate in general only under the dihedral group of order $2d$. If however $d = 3$, it may be self-conjugate under a group of order 18, under which all six permutations are made on the vertices of its fixed triangle (Theorem 3). If therefore $d \neq 3$, there will be conjugate with this cyclic group $\Omega/2d$ cyclic groups, where Ω denotes the order of the whole group. In these groups there will be $(d-1)\Omega/2d$ transformations excluding the identity. If $d = 3$, there will be conjugate with the cyclic group either $\Omega/2\cdot 3$ or $\Omega/6\cdot 3$ cyclic groups containing

$$(3-1)\frac{\Omega}{2\cdot 3} \quad \text{or} \quad (3-1)\frac{\Omega}{6\cdot 3}$$

transformations other than the identity. If two o-lines are not conjugate, the two cyclic groups which leave them fixed will not be conjugate. Corresponding then to each conjugate set of o-lines on which the group is dihedral there will be a conjugate set of cyclic groups of type I.

Any two o-lines through the center of a reflection which are conjugate under the whole group are conjugate also under the group commutative with that reflection. For if two o-lines, b and b', through A are conjugate, there are transformations which send b to b' and a center on b into any center on b'. In particular there are transformations which leave A fixed and send b to b'. Hence if the group commutative with a reflection be of order g, there will be $\frac{1}{2}g$ o-lines in each conjugate set through its center.

Theorem 10. *In a multiplicative group which does not contain homologies of period 3 but which contains a C_3 self-conjugate under a G_9 permuting cyclically the vertices of its fixed triangle, any two of the C_3 which are conjugate under the whole group are conjugate also under the subgroup leaving the G_9 invariant.*

We choose two of the C_3 in the G_9 as those generated by $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$, where $\omega^2 + \omega + 1 = 0$. The four fixed triangles of the C_3 in the G_9 then form the Hessian configuration. All the G_9 in which one of these C_3 lies form a single conjugate set. For if the group leaving its fixed triangle point-wise invariant is of order d, it will lie in $d/3$ groups G_9. But the group leaving the triangle point-wise invariant which also leaves one of the G_9 invariant is of order 3, since we have assumed no homologies of period 3 to be present. Hence the $d/3$ groups G_9 are all conjugate.

If there are transformations in the group which transform one C_3 into another C_3 in the same G_9, there will then be transformations which transform the first C_3 into the second, and a G_9 in which the first lies into any G_9 in

which the second lies. In particular the two C_3 will be conjugate under the group which leaves invariant the G_9 in which both lie.

§ 8. Multiplicative Groups containing no Four-Groups.

Theorem 11. *The only multiplicative groups, which do not leave invariant a point, line, or triangle, and which contain reflections but no four-groups, are the two Hessian groups G_{36} and G_{72}.*

If a group contains reflections but no four-groups, all the centers of the reflections will lie on o-lines through one of them. Consequently all reflections in the group will be conjugate. Commutative with each reflection there will be a certain group, whose order we denote by g. No transformation except the identity can be commutative with more than one of the reflections. If Ω denotes the order of the whole group, there will then be Ω/g reflections and $(g-1)\Omega/g$ transformations other than the identity in the groups commutative with the reflections. We suppose there are r conjugate sets of o-lines on which the centers of the reflections lie. The order of the group will then be equal to the product of the order of the group commutative with a single reflection times the number of reflections. Hence

$$\Omega = g\left\{1 + \frac{g}{2}\sum_{i=1}^{r}(d_i - 1)\right\}.$$

The order of the group will also be equal to the number of transformations which it contains, as follows:

$$\Omega = 1 + (g-1)\frac{\Omega}{g} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} + \cdots,$$

where, if $d_i \neq 3, f_i = 2$; if $d_i = 3, f_i = 2, 6$.

We find that there are the following four solutions:

$$g = 2,\ r = 1,\ f_1 = 2,\ \Omega = 2d.$$

$$g = 2,\ r = 4,\ d_i = 3, f_i = 6,\ \Omega = 18.$$

$$g = 4,\ r = 2,\ d_i = 3, f_i = 6,\ \Omega = 36.$$

$$g = 8,\ r = 1,\ d_1 = 3, f_1 = 6,\ \Omega = 72.$$

The first solution gives simply a single dihedral group. In each of the other three cases there are four C_3, each of which is invariant under a G_{18}. If two C_3 be $[x_1, \omega x_2, \omega^2 x_3]$, $[x_2, x_3, x_1]$, where $\omega^2 + \omega + 1 = 0$, the four C_3 are then determined. If these groups exist they must then be subgroups of the Hessian G_{216}. They do exist, since the G_{216} is readily shown to be isomorphic with a tetrahedral group permuting the four triangles.

§ 9. Multiplicative Groups containing Four-Groups.

Theorem 12. *If a multiplicative group contains four-groups, every dihedral group which it contains will be contained by a group leaving invariant a conic.*

We choose two reflections which generate the dihedral group as those with centers (001) and (011), and axes $x_3 = 0$ and $x_2 + \lambda x_3 = 0$ respectively. The invariant family of conics is then $Ax_1^2 + x_2^2 + \lambda x_3^2 = 0$, where A is the parameter (Fig. 3).

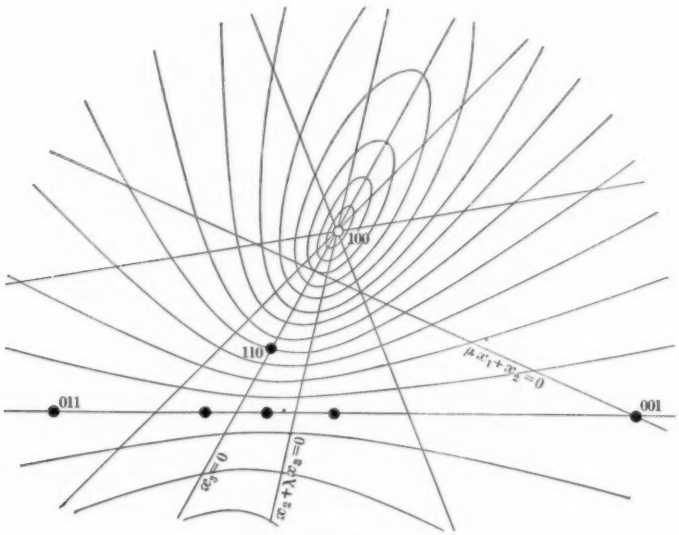

Fig. 3.

If there is a single reflection which does not lie in a four-group, the centers of all the other reflections will lie on o-lines through its center. Hence all reflections will be conjugate and no reflection can lie in a four-group. If four-groups do appear therefore, every reflection must lie in a four-group. Consider for example the reflection with center (001) and axis $x_3 = 0$, and suppose that the only reflections with which it is commutative have centers at (100) and (010). Since no triangle is supposed left invariant, (100) will be conjugate with points not on $x_1 = 0$ or $x_2 = 0$. These points must then lie on o-lines through (001), and hence must be conjugate with (001). But if (100) is a center of a reflection, more than two reflections will be commutative with that reflection. Hence more than two reflections in such a case must be commutative with the reflection with center (001).

There will be then in any case reflections commutative with that with center (001), whose centers are not (100) or (010). If the center of such a reflection

is (110) and its axis is $\mu x_1 + x_2 = 0$, the reflection will leave invariant in common with the dihedral group the conic $\mu x_1^2 + x_2^2 + \lambda x_3^2 = 0$.

Theorem 13. *A group which does not leave invariant a point, line, or triangle, and which contains four-groups, will contain either an octahedral group G_{24} or an icosahedral group G_{60}.*

Any group which does not leave invariant a point, line, or triangle, and which contains reflections, will contain two reflections which are not commutative and hence a dihedral group. It will then contain a group leaving invariant a conic (Theorem 12). The groups which leave a conic invariant are the regular solid groups. The only two which contain dihedral groups as subgroups are the octahedral group G_{24}, and the icosahedral group G_{60}.

Theorem 14. *The G_{168}^7 is the only multiplicative group which contains an octahedral group G_{24}, but not an icosahedral group G_{60}.*

If we choose the four points which are permuted by the octahedral group as (111), (−111), (1−11), (11−1) (Fig. 4), it may be generated by the following transformations:

$$S\colon\ [-x_1, x_3, x_2],$$

$$T\colon\ [x_2, x_1, x_3],$$

$$U\colon\ [x_1, x_3, x_2].$$

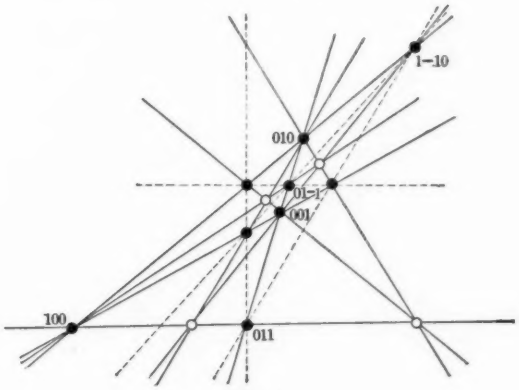

FIG. 4.

These generators satisfy the relations:

$$S^2 = T^2 = U^2 = (ST)^3 = (SU)^2 = (TU)^3 = I.$$

The G_{24} contains two sets of reflections, three in one set conjugate with SU and six in the other set conjugate with U. Any group which contains the G_{24} must contain reflections not in the G_{24}, since the latter can be invariant under

no larger group. Every reflection not commutative with one of the three reflections conjugate with SU must generate with the four-group containing those three reflections an octahedral group, since it will leave invariant a conic common to that four-group. There can be no dihedral group of order greater than 8 (Theorem 12). Hence each of the reflections conjugate with SU can be commutative with but four reflections. The invariant four-group of the G_{24} generated by S, T, and U, i. e., the four-group with fixed triangle (100)(010)(001), can therefore lie in but three other octahedral groups, i. e., groups having for fixed triangles (100)(011)(01–1), (010)(101)(10–1), (001)(110)(1–10). Hence every reflection in the group containing the G_{24} must be commutative with a reflection of the G_{24}. No more reflections can be commutative with a reflection which is conjugate under the G_{24} with SU, and but two more reflections can be commutative with each of the six reflections which are conjugate with U under that group. The only group which may contain the G_{24} is therefore a group containing $3 + 6 + 6\cdot 2 = 21$ reflections. The centers of these twenty-one reflections must lie on the four axes and on four o-lines through a center of one of them. They will all be conjugate, and the order of the group will be $8\cdot 21 = 168$.

A reflection commutative with U must generate with S a dihedral group of order 8 (Theorem 12). Such a reflection is

$$V\colon \begin{aligned} x_1' &= -\lambda^2 x_2 - \lambda^2 x_3, \\ x_2' &= -2x_1 + \lambda x_2 - \lambda x_3, \\ x_3' &= -2x_1 - \lambda x_2 + \lambda x_3. \end{aligned}$$

This reflection must generate a G_{24} with the dihedral G_6 generated by T and U, and hence either its product by T must be of period 3 or the product of VU by T must be of period 3. Without loss of generality we may take $(TV)^3 = I$. The condition for this is $\lambda^2 - \lambda + 2 = 0$, $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{-7}$. In this case we have the generational relations:

$$V^2 = (VU)^2 = (VT)^3 = I, \qquad VSVS = U.$$

A group of order 168 is then generated by S, T, U, and V. We suppose the operators of the G_{24} written down in a horizontal row, and form a multiplication table (multiplying on the left) of seven rows, of which the multipliers are I, V, TV, UTV, STV, $SUTV$, $TSUTV$. If these seven rows be numbered from 1 to 7 in the order in which their multipliers are written, they are permuted as follows by the four generators when applied as left-hand multipliers:

S: (1)(2)(35)(46)(7),

T: (1)(23)(4)(5)(67),

U: (1)(2)(34)(56)(7),

V: (12)(3)(4)(56)(7).

The 168 operators are thus permuted among themselves. The rows must all be distinct, for if two coincide, all must coincide. This is impossible, since V does not lie in the octahedral group. Hence a group of order 168 is generated. It is representable on seven letters, e. g., on the seven rows, and is simple.

Theorem 15. *The G^{6}_{360} is the only multiplicative group containing an icosahedral group G_{60}.*

We choose (100), (010), (001) as the three centers of one of the four-groups of a G_{60}. Since a four-group in a G_{60} is self-conjugate under a tetrahedral group, there will be four C_3 permuting cyclically these three centers. Each of the C_3 must be self-conjugate under a G_6. We choose as (111) the fixed point of a C_3 through which pass the three axes of reflections. Two of the transformations of the tetrahedral group are then

$$E_1\colon [x_2, x_3, x_1], \qquad E_2\colon [x_1, -x_2, -x_3].$$

A reflection generating with E_1 a dihedral G_6 takes the form

$$E_3\colon \begin{aligned} x_1' &= (\alpha^2+\alpha)x_1 - \alpha x_2 + (\alpha+1)x_3,\\ x_2' &= -\alpha x_1 + (\alpha+1)x_2 + (\alpha^2+\alpha)x_3,\\ x_3' &= (\alpha+1)x_1 + (\alpha^2+\alpha)x_2 - \alpha x_3. \end{aligned}$$

The centers of the fifteen reflections in the group with the exception of (010) and (001) must lie on o-lines through (100) which contain three or five centers (Fig. 5). At least one of the three reflections in the dihedral group containing

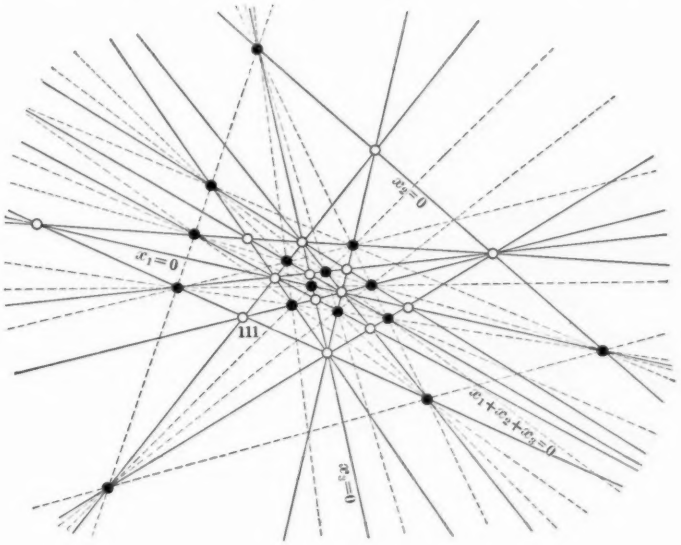

FIG. 5.

E_1 must be such that its product by E_2 is of period 3. In order that $(E_2E_3)^3 = I$, we must have $\alpha^2 + \alpha - 1 = 0$, $\alpha = (-1 \pm \sqrt{5})/2$. In this case the three transformations satisfy the following relations:

$$E_1^3 = E_2^2 = E_3^2 = (E_1E_2)^3 = (E_1E_3)^2 = (E_2E_3)^3 = I.$$

They therefore generate the icosahedral group G_{60}.*

There is a single conjugate set of G_{60} under the whole collineation group, since the two G_{60} which correspond to the two values of α are conjugate under the transformation $[x_1, x_3, x_2]$.

We inquire first whether any group can contain the G_{60} and have only two reflections commutative with each reflection. The reflections in such a group must arrange themselves by threes in four-groups, the vertices of the fixed triangles of these four-groups being permuted cyclically. Every reflection not in the four-group with fixed triangle (100)(010)(001) must generate with that four-group a G_{60}. Corresponding then to each pair of o-lines through one center which contain 3 centers there will be a pair of o-lines which contain 5 centers. Since there is no C_4 commutative with a reflection, the group cannot contain as a subgroup either the Hessian G_{36} or the G_{72}. Consequently since there can be but one conjugate set of o-lines containing 3 centers, the vertices of the fixed triangles of the C_3 cannot be permuted cyclically (Theorem 10).

If we denote by Ω the order of the whole group and by g the order of the group commutative with a four-group, Ω must then satisfy a Diophantine equation of the form:

$$\Omega = 1 + (g-1)\frac{\Omega}{3g} + (5-1)\frac{\Omega}{2\cdot 5} + (3-1)\frac{\Omega}{2\cdot 3} + \cdots.$$

In order that the coefficient of Ω on the right shall be less than unity, we must have $g = 4$, $\Omega = 60$. Hence no group can contain the G_{60} if only two reflections are commutative with a reflection.

We suppose next that a reflection is commutative with more than two reflections. Since the group cannot contain a dihedral group of order greater than 10 (Theorem 12), the centers and axes of two reflections which are commutative with the same reflection, but not with each other, must separate each other harmonically. There can then be but four reflections commutative with any reflection. A four-group can then be commutative only with itself. A reflection commutative with E_2 must then generate a G_{24} with the G_{12} generated by E_1 and E_2. Such a G_{24} is an octahedral group and will leave invariant a conic. This conic cannot be $x_1^2 + x_2^2 + x_3^2 = 0$, since that is left invariant by a G_{60}. It must then be $x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = 0$, where ω is a cube root of unity. There must be at least 45 reflections in the group, since a reflection not in the G_{60}

* E. H. Moore, Proceedings of the London Mathematical Society, vol. 28 (1897), pp. 357–366; Dickson, *Linear Groups*, p. 289.

must be conjugate with at least 30 reflections under that group. We may show that the maximum number must be 45. There are three reflections in the invariant four-group of the G_{12} generated by E_1 and E_2, and two more reflections commutative with each of those three. Every reflection not commutative with any one of the three reflections of the four-group will leave a conic invariant in common with the four-group, and hence must generate with the four-group either a G_{24} or a G_{60}. The four-group can lie in but three octahedral groups other than the one under which it remains invariant, i. e., octahedral groups whose invariant four-groups contain a reflection in common with the above four-group. Each of the three G_{24} will contain four reflections not commutative with any one of three reflections of that four-group. Since the four-group can be commutative only with itself, every G_{60} in which it lies must contain the G_{12} generated by E_1 and E_2. Since the conic $x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = 0$ is left invariant by a G_{24}, there can be but two such G_{60}, i. e., those having for fixed conics $x_1^2 + x_2^2 + x_3^2 = 0$, $x_1^2 + \omega^2 x_2^2 + \omega x_3^2 = 0$. There can then be but $3 + 3\cdot 2 + 3\cdot 4 + 2\cdot 12 = 45$ reflections. There will be four o-lines through the center of a reflection, on which lie 5 centers. Hence the group commutative with a reflection must be of order 8 and the whole group of order $8\cdot 45 = 360$. One of the reflections commutative with E_2 is

$$E_4 : \ [x_1, \ \omega^2 x_3, \ \omega x_2].$$

This reflection satisfies with the three generators of the G_{60} the following relations:

$$E_4^2 = (E_1 E_4)^2 = (E_2 E_4)^2 = (E_3 E_4)^3 = I.$$

It therefore generates with the G_{60} a G_{360}^6.* Two G_{360}^6 corresponding to the choice of ω contain the G_{60}. The G_{360} however contains two sets of G_{60} and a transformation carrying a G_{60} not conjugate with the G_{60} generated by E_1, E_2, E_3 into the latter transforms one G_{360} into the other. There is therefore a single conjugate set of G_{360} in the plane.

§ 10. Groups containing only Transformations of Type I.

A complete discussion of multiplicative groups which contain homologies has been given. Any group which contains only transformations of type I must be of odd order, since a transformation of period 2 is a reflection. Any cyclic group contained by such a group can then be self-conjugate only under itself or under a group permuting cyclically the vertices of its fixed triangle. If Ω denotes the order of the whole group, it must then satisfy a Diophantine equation of the form

$$\Omega = 1 + \sum_{i=1}^{r} (d_i - 1) \frac{\Omega}{f_i d_i} \qquad (f_i = 1, 3).$$

* Moore, loc. cit., Dickson, loc. cit.

If the order of the group is divisible by 3, we may take d_1 to be divisible by 3. If then $f_1 = 3$, there must be four conjugate sets of C_3 (Theorem 10). Hence $r \geqq 4$, and each of the first four d's is divisible by 3. This can happen only if $r = 4$, $d_2 = d_3 = d_4 = 3$, $\Omega = 3d_1$. If on the other hand $f_1 = 1$, we find $r = 1$, $\Omega = d_1$, or $r = 2$, $d_1 = 3$, $\Omega = 3d_2$.

If the order of the group is not divisible by 3, we have $r = 1$, $\Omega = d_1$. We have therefore

Theorem 16. *No multiplicative group which does not leave invariant a point, line or triangle can contain only transformations of type I.*

The discussion of the groups in the ordinary plane is therefore complete.

§ 11. Groups containing Transformations of Type III, but not Elations.

We now consider the possibility of groups containing transformations of type III, but not elations, and consequently no transformations of type II. There can be no homologies present which have a common center but different axes, or vice-versa, so that Theorem 4 holds. Theorem 5 does not hold, but instead we have

Theorem 17. *A group containing transformations of type III but not elations cannot contain two homologies such that the line joining their centers passes through the intersection of their axes unless they are both of period* 2.

The product of two homologies such that the line joining their centers passes through the intersection of their axes will leave invariant two points on that line unless one transformation on that line is conjugate with the inverse of the other. Unless the homologies are both of period 2 we may choose their powers so that this is not the case. The product is therefore of type II in the plane.

Theorems 6, 7, 8, 9 are proved independently of the assumption of the non-existence of transformations of type III in the group. The Hessian group G_{216} however contains no transformations of type III. Hence we have

Theorem 18. *There is no group which contains transformations of type III but not elations, which does not leave invariant a point, line, or triangle, and which contains homologies of higher period than* 2.

If transformations of type III appear, there may be a new kind of o-lines through the center of a reflection, i. e., o-lines on which the group of the points is metacyclic, containing an additive base. There cannot be more than a single conjugate set of such o-lines, since their number must be of the form $1 + fp^m$, where p^m denotes the order of the additive group on the line. If the order of the group commutative with a reflection is g, and the order of the subgroup of that group which leaves fixed an o-line of this sort is h, the number of such o-lines through the center of the reflection will be g/h.

Theorem 10 holds in this case also. In the proof of Theorem 11 however it

is necessary to consider the additional case where o-lines of the sort described above appear.

If there are no reflections commutative with a reflection, and if there is a set of o-lines, each of which is left invariant by a metacyclic group of order hp^m, any group which exists (of order Ω) will contain $(g-1)\Omega/g$ transformations in the groups commutative with the reflections and $(p^m-1)\Omega/hp^m$ transformations of type III. Hence Ω must satisfy an equation of the form:

$$\Omega = 1 + (g-1)\frac{\Omega}{g} + (p^m-1)\frac{\Omega}{hp^m} + \cdots.$$

In order that the coefficient of Ω on the right shall be less than unity, no more terms representing transformations can appear, and also $h = g$. We then have $\Omega = gp^m$, which represents the single metacyclic group.

If we combine this result with Theorem 11, and observe that the G_{36} and G_{72} contain no transformations of type III, we have

Theorem 19. *No group exists which contains transformations of type III but not elations, which does not leave invariant a point, line, or triangle, and which contains reflections but no four-groups.*

Theorem 12 holds in this case without modification. We prove also

Theorem 20. *If a group containing four-groups contains a metacyclic group of order $2p^m$ (the operators of period p being of type III), that metacyclic group will be contained by a group leaving invariant a conic.*

A metacyclic group of order $2p^m$ will leave invariant a one-parameter family of conics. For consider a single transformation of type III, such as

$$[x_1 + x_2 + x_3,\ x_2 + 2x_3,\ x_3].$$

This leaves invariant the family $x_2^2 + Ax_3^2 - 4x_1x_3 = 0$ (Fig. 6). The above

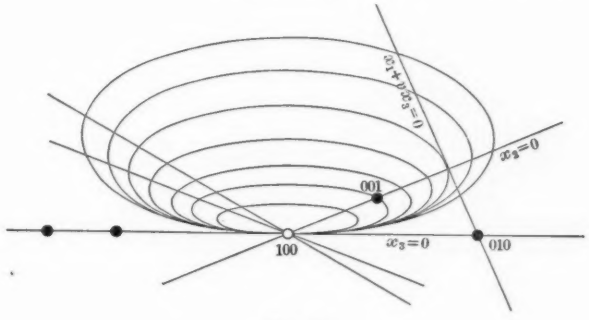

FIG. 6.

transformation is invariant under any transformation of type III leaving fixed (100) and $x_3 = 0$ and the same family of conics. Hence we may choose the

center of a reflection in the metacyclic group arbitrarily as (010). The axis will be a line, $x_2 + \mu x_3 = 0$, passing through (100). This reflection will transform the above transformation into a transformation which is its inverse on $x_3 = 0$. The transformed transformation is

$$[x_1 - x_2 + (1 - 2\mu)x_3,\ x_2 - 2x_3,\ x_3].$$

The product of the two is an elation unless $\mu = 0$. The reflection must therefore leave fixed the conics. Hence every reflection in the metacyclic group leaves fixed the conics. Since the metacyclic group of order $2p^m$ may be generated by the p^m reflections which it contains, it leaves fixed the conics.

Since we have assumed the existence of four-groups, the reflection with center (010) and axis $x_2 = 0$ will lie in a four-group. Since the metacyclic group is invariant under any elation with center (100) and axis $x_3 = 0$, we may choose the center of a reflection in that four-group arbitrarily as (001). Its axis will be a line, $x_1 + \nu x_3 = 0$, passing through (010). This reflection leaves invariant in common with the metacyclic group the conic $x_2^2 - 2\nu x_3^2 - 4x_1 x_3 = 0$.

Theorem 21. *If a group containing four-groups contains transformations of type III, it will contain a group leaving invariant a conic, which is either an octahedral group G_{24}, an icosahedral group G_{60}, a group of order $(p^m + 1)p^m(p^m - 1)/2$, or a group of order $(p^m + 1)p^m(p^m - 1)$.*

Any group which contains reflections will contain two reflections which are not commutative, provided no point, line or triangle is left invariant. Hence it will contain either a dihedral group or a metacyclic group and therefore (either by Theorem 12 or by Theorem 20) a group leaving invariant a conic. The group leaving a conic invariant is simply isomorphic with the group of the points on a line.* Hence this group must be either an octahedral group G_{24}, an icosahedral group G_{60}, a group of order $(p^m + 1)p^m(p^m - 1)/2$, or a group of order $(p^m + 1)p^m(p^m - 1)$. (See § 1.)

Theorem 22. *A $G^7_{168}(p = 3)$ is the only group containing transformations of type III such that its largest subgroup leaving invariant a conic is a G_{24}.*

The proof of this theorem is the same as that of Theorem 14; except that, having obtained the group of order 168, we have yet to determine when it contains transformations of type III. This will be the case only if $p = 3, 7$. For $p = 7$ the G_{168} leaves invariant a single conic. Hence the theorem is as stated.

Theorem 23. *A $G^6_{360}(p = 5)$ and a $G_{720}(p = 5)$ are the only groups containing transformations of type III such that their largest subgroup leaving invariant a conic is a G_{60}.*

The proof is the same as that of Theorem 15 with slight modifications. We make use of Theorem 20 as well as of Theorem 12 in showing that the only

* VEBLEN and YOUNG, *Projective Geometry*, Chapter VIII, Theorem 15.

o-lines which exist contain 3 or 5 centers. We show in the same way that there can be but 45 reflections in the group. It was shown under Theorem 15 that the group commutative with a reflection was of order 8, since there must be through its center four o-lines containing 5 centers each. For $p = 5$ however it is possible that there may also be a group of order 16 permuting these four o-lines. A transformation of period 8 transforming the $G_{360}(p = 5)$ into itself is given by

$$[x_1,\ (2\omega + 1)x_2 + (\omega - 1)x_3,\ -(\omega + 2)x_2 + (2\omega + 1)x_3].$$

A group of order 720 therefore exists.

The G_{360} also contains transformations of type III if $p = 3$. In this case however it leaves invariant a single conic. Hence the theorem is as stated.

§ 12. Groups containing Transformations of Type III which contain larger Subgroups leaving a Conic Invariant.

Theorem 24. *There exists no group which does not contain elations or homologies of higher period than* 2, *and which contains a set of reflections each of which is commutative with a group* (2, 1) *isomorphic with either the tetrahedral, octahedral, or icosahedral group.*

If a group commutative with a reflection is (2, 1) isomorphic with either the tetrahedral or icosahedral group, it can contain no four-group. For all the involutions in either a G_{12} or a G_{60} lie in four-groups (on the axis), in any one of which all three involutions are conjugate. If two of the involutions are performed by reflections, the third will be performed by a C_4. But this is impossible, since the three involutions are conjugate. Hence no one of the involutions on the line can be performed by a reflection.

If the group commutative with a reflection is (2, 1) isomorphic with an octahedral group, none of the involutions of the included tetrahedral group can be performed by reflections. The six involutions not in the tetrahedral group however may be performed by reflections. In this case there will be twelve reflections commutative with the given reflection. The group commutative with the given reflection will contain four conjugate dihedral groups of order 12 and three conjugate dihedral groups of order 8.

Each of these dihedral groups will be contained by a group leaving invariant a conic (Theorem 12). Hence some of the twelve reflections commutative with the given reflection will be conjugate with that reflection, and hence all will be. There will then be twelve reflections commutative with every reflection. A dihedral group of order 8 will therefore lie in a group leaving invariant a conic which contains two conjugate sets of reflections. Such a group will be of order $(p^m + 1)p^m(p^m - 1)$, where $2(p^m - 1) \doteq 8$, $p^m = 5$. No group is then possible unless $p = 5$.

Consider now a dihedral G_{12}. Each reflection commutative with one of the six reflections of this G_{12} (other than the invariant reflection) must generate a G_{120} with the G_{12}. Since a G_{120} contains 25 reflections and the G_{12} contains 7, the reflections commutative with these six reflections which are not in the dihedral group must be grouped in sets of 18. But there are 60 such reflections and hence this is impossible. No group can then exist for $p = 5$.

Theorem 25. *A G^7_{2520} is the only group such that its largest subgroup leaving invariant a conic is a G_{120} $(p = 5)$, provided that no elations are present.*

A G_{120} $(p = 5)$ is generated by

$$E_1\colon [x_2, x_3, x_1], \qquad E_2\colon [x_1, -x_2, -x_3], \qquad E_5\colon [x_1, x_3, x_2],$$

$$E_3\colon x_1' = x_1 - 2x_2 - 2x_3, \qquad x_2' = -2x_1 - 2x_2 + x_3, \qquad x_3' = -2x_1 + x_2 - 2x_3.$$

The invariant conic is $x_1^2 + x_2^2 + x_3^2 = 0$. The G_{120} contains two conjugate sets of reflections. Each reflection conjugate with E_2 is commutative with a dihedral group of order 8; each reflection conjugate with E_5 is commutative with a dihedral group of order 12. (Fig. 7.)

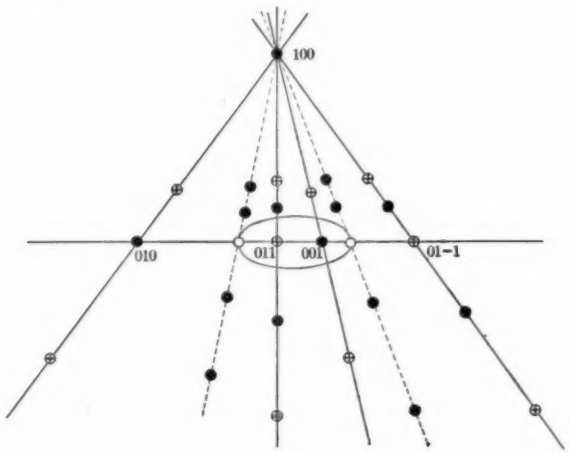

FIG. 7.

A group which contains the G_{120} will contain reflections not in the G_{120}. Since the G_{120} is supposed the largest subgroup leaving invariant a conic, no dihedral group of order greater than 12 can appear (Theorem 12). It may be shown readily that under any group containing the G_{120} E_2 must be invariant under a larger group. The only possible larger group which will not introduce a dihedral group of order greater than 12 and which will allow the group of the

points on $x_1 = 0$ and $x_1 - x_3 = 0$ to be dihedral is a G_{24} having for fixed triangle the fixed triangle of the G_{12} generated by E_1 and E_2. This G_{24} contains the C_3:

$$[x_1,\ \omega x_2,\ \omega^2 x_3] \qquad (\omega^2+\omega+1=0).$$

The fixed triangles of all the C_3 in the group will then be invariant under a G_9. The four-group which is invariant under the G_{12} generated by E_1 and E_2 will then lie in three G_{120}. Each of those G_{120} will contain 22 reflections not in that four-group. The four-group will also lie in nine G_{24} under which it is not invariant, i. e., G_{24} whose invariant four-groups have each a reflection in common with the above four-group. Each of these nine G_{24} will contain 4 reflections not in any one of the three G_{120}. There will then be $3+3\cdot 22+9\cdot 4=105$ reflections. These will all be conjugate and the order of the group must be $24\cdot 105 = 2520$.

As a product of the above C_3 and E_5 we obtain

$$E_4\colon\ [x_1,\ \omega^2 x_3,\ \omega x_2].$$

The five generators are found to satisfy the relations:

$$E_1^3 = E_2^2 = E_3^2 = E_4^2 = E_5^2 = I, \qquad (E_iE_{i+1})^3 = I, \qquad (E_iE_j)^2 = I \quad (j>i+1).$$

A G_{2520}^7 is then generated.*

Theorem 26. *No group which does not contain elations can contain a group of order* $(p^m+1)p^m(p^m-1)/2$ *or* $(p^m+1)p^m(p^m-1)$ *leaving a conic invariant, provided* $p^m>5$, *and provided it contains no larger group leaving a conic invariant.*

The group commutative with a reflection in a group of order

$$\tfrac{1}{2}(p^m+1)p^m(p^m-1) \quad \text{or} \quad (p^m+1)p^m(p^m-1)$$

leaving a conic invariant is dihedral and hence the group of the points on the axis of the reflection is dihedral. There is but one pair of points on the axis which are interchanged provided the group of the points on the axis is larger than the four-group, i. e., provided the group commutative with the reflection is of order greater than 8. For the group of order $(p^m+1)p^m(p^m-1)/2$ this will be true if $p^m\pm 1>8$, $p^m>9$; for the group of order $(p^m+1)p^m(p^m-1)$ it will be true if $2(p^m\pm 1)>8$, $p^m>5$. If then $p^m>5$ and if the group is not the $G_{168}(p^m=7)$ or the $G_{360}(p^m=3^2)$, there will be but one pair of points on the axis which are interchanged by the group.

Excluding for the moment these two cases, we may show that there can be no more reflections commutative with a reflection. For there certainly cannot

* MOORE, l. c.; DICKSON, l. c.

be more than two such reflections, i. e., the two leaving the two points pointwise invariant. (If a reflection interchanges the points, it will leave fixed the conic.) But if we add two reflections we must add more than two, since any such reflection must generate together with a dihedral group containing the reflection with which it is commutative (other than the dihedral group which leaves the latter reflection invariant) a group of order $(p^m+1)p^m(p^m-1)/2$ or $(p^m+1)p^m(p^m-1)$ leaving invariant a conic. This group can have in common with the other group which leaves fixed a conic only the reflections of the dihedral group. Hence there will be more than two reflections added which are commutative with the latter reflection.

If the group is the $G_{168}(p^m=7)$ or the $G_{360}(p^m=3^2)$, there can be no more reflections commutative with a reflection. For since the group is supposed the largest group present which leaves invariant a conic, there can be no dihedral group present of order greater than 10 (Theorem 12). Hence there can be only four reflections commutative with any reflection in the group.

Suppose now that there is a reflection not commutative with any reflection in the group. It will leave invariant in common with any four-group of that group a conic. It must generate with that four-group some group leaving invariant a conic which can contain no reflections commutative with any of the three reflections of the four-group except those three reflections themselves. Such a group must be an icosahedral group G_{60}. Every reflection not in the original group leaving fixed a conic must then generate with this four-group a G_{60}. But there cannot be any transformations commutative with the four-group other than those of the four-group itself without involving more reflections commutative with the reflections of the four-group. Hence there cannot be more than four C_3 permuting cyclically the vertices of the fixed triangle of the four-group, i. e., all the G_{60} must contain the same tetrahedral group. A tetrahedral group however leaves fixed but three conics, one of which is the original conic. There can then be but two such G_{60}, each of which contains twelve reflections other than those of the four-group. There cannot then be more than 24 more reflections, which is impossible.

§ 13. Groups containing only Transformations of Types III and I.

Theorem 27. *There exists no group which contains only transformations of types* III *and* I *and which does not leave invariant a point, line, or triangle.*

Any group which contains only transformations of types III and I must be of odd order, since a transformation of period 2 is a reflection.

An additive group of order p^m containing transformations of type III will be self-conjugate under a metacyclic group of order d_1p^m, containing p^m conjugate cyclic groups of order d_1 which contain transformations of type I. (In partic-

ular d_1 may be unity.) Any cyclic group of order d_i containing transformations of type I will be self-conjugate either under itself only or under a group of order $3d_i$ permuting cyclically the vertices of its fixed triangle.

If we denote the order of the whole group by Ω, and enumerate the transformations which the group must contain, we are led to the following Diophantine equation:

$$\Omega = 1 + (p^m - 1)\frac{\Omega}{d_1 p^m} + \sum_{i=1}^{r}(d_i - 1)\frac{\Omega}{f_i d_i} \qquad (f_i = 1, 3).$$

If $f_1 = 3$, we may show that $d_1 = 3$. Let the fixed point and line of an additive group be (100) and $x_3 = 0$ and let the vertices of the fixed triangle of a cyclic group of order d_1 in the metacyclic group, under which the additive group is self-conjugate, be (100), (010), (001). A transformation of period d_1 in that cyclic group will be given by

$$[x_1, \omega x_2, \omega^e x_3], \tag{1}$$

where the period of ω is d_1. A transformation of type III in the additive group will be given by

$$\begin{aligned} x_1' &= x_1 + \alpha x_2 + \beta x_3, \\ x_2' &= \quad x_2 + \gamma x_3, \qquad (\alpha \neq 0, \gamma \neq 0). \\ x_3' &= \quad x_3 \end{aligned} \tag{2}$$

If we transform (2) by (1), we obtain

$$\begin{aligned} x_1' &= x_1 + \alpha\omega x_2 + \beta\omega^e x_3, \\ x_2' &= \quad x_2 + \gamma\omega^{e-1} x_3, \\ x_3' &= \quad x_3. \end{aligned} \tag{3}$$

If we transform (2) by (3), we obtain

$$\begin{aligned} x_1' &= x_1 + \alpha x_2 + (\beta + \alpha\gamma\omega^{e-1} - \alpha\gamma\omega)x_3, \\ x_2' &= \quad x_2 \quad + \quad \gamma x_3, \\ x_3' &= \quad x_3. \end{aligned} \tag{4}$$

The transformation (4) is the same as (2) on the line $x_3 = 0$. Hence (4) must be identical with (2). Hence $\omega^{e-1} = \omega$, $e \equiv 2 \pmod{d_1}$. But if the vertices of the fixed triangle of (1) are permuted cyclically, we must have $e^2 - e + 1 \equiv 0 \pmod{d_1}$. [See the discussion under Theorem 2.] Hence $2^2 - 2 + 1 \equiv 0 \pmod{d_1}$; $d_1 = 3$.

But if $f_1 = 3$, $d_1 = 3$, there must be four conjugate sets of C_3 (Theorem 10).

Hence $r \geqq 4$. The coefficient of Ω on the right is then greater than unity, and hence there can be no solution.

If $f_1 = 1$, $r = 1$, $\Omega = d_1 p^m$. This represents a single metacyclic group.

If $d_1 = 1$, either $r = 2$, $f_2 = 3$, $p^m = 3$, $\Omega = 3d_2$, or else $r = 1$, $\Omega = p^m$. The first represents a group permuting cyclically the vertices of a triangle, and the second a single additive group.

§ 14. Groups containing Elations.

Theorem 28. *The only groups which do not leave invariant a point or a line and which contain elations are: the* $HO(3, p^{2k})$*; groups containing the* $HO(3, p^{2k})$ *as self-conjugate subgroups of index* 3, *if* $p^k + 1$ *is divisible by* 3*; the* $LF(3, p^k)$*; groups containing the* $LF(3, p^k)$ *as self-conjugate subgroups of index* 3, *if* $p^k - 1$ *is divisible by* 3.

Any group which contains elations will contain a group of largest order consisting wholly of elations with a common axis and center. We denote the order of such a group by p^k. We choose the center and axis of one such group as (010) and $x_3 = 0$. Since we assume that no point or line remains invariant under the group, there will be groups of elations whose centers do not lie on $x_3 = 0$ and whose axes do not pass through (010). We choose the center and axis of such a group as (001) and $x_2 = 0$. By hypothesis then the order of the group consisting of all the elations with center (001) and axis $x_2 = 0$ will be less than or equal to p^k. The group generated by these two groups of elations will leave invariant (100) and $x_1 = 0$. The group of the points on $x_1 = 0$ cannot contain an additive group of higher order than p^k, and hence (by § 1) must be of order $(p^k + 1)p^k(p^k - 1)/2$ or $60 (p^k = 3)$. (The group of order $(p^k + 1)p^k(p^k - 1)$ on the line cannot be generated by additive transformations only.)

Under either of these two groups, (010) and $x_3 = 0$ will be conjugate with (001) and $x_2 = 0$. Hence (010) and $x_3 = 0$ will be conjugate with the center and axis of any group of elations, since in any conjugate set of such groups there will be groups whose centers do not lie on $x_3 = 0$ and whose axes do not pass through (010).

If the group of the points on $x_1 = 0$ is of order $(p^k + 1)p^k(p^k - 1)/2$, we may choose it as the group generated by the two groups of elations

$$(E): [x_1, x_2 + \lambda x_3, x_3], \qquad [x_1, x_2, \lambda x_2 + x_3],$$

where λ takes all values in the $GF(p^k)$. This group contains an invariant reflection with center (100) and axis $x_1 = 0$ (Theorem 1). The group contains a set of cyclic groups of period $(p^k - 1)/2$ on $x_1 = 0$. One such group is generated by $[x_1, \eta x_2, \eta^{-1} x_3]$, where η is of period $p^k - 1$. This cyclic group is of

period $p^k - 1$ on $x_2 = 0$ and $x_3 = 0$. The group generated by the two groups of elations (E) also contains a set of cyclic groups of period $(p^k + 1)/2$ on $x_1 = 0$. Such a cyclic group which leaves fixed $(01I)$ and $(01{-}I)$, where I is a square-root of a not-square in the $GF(p^k)$, may be generated by a transformation of the form

$$x_1' = x_1, \qquad Ix_2' + x_3' = J(Ix_2 + x_3), \qquad Ix_2' - x_3' = J^{-1}(Ix_2 - x_3),$$

where J is a mark of period $p^k + 1$. This cyclic group is of period $p^k + 1$ on $Ix_2 + x_3 = 0$ and $Ix_2 - x_3 = 0$.

A group of order $60\,(p^k = 3)$ on $x_1 = 0$ is generated by the two groups of elations:

$$[x_1,\ x_2 + x_3,\ x_3], \qquad [x_1,\ x_2,\ ix_2 + x_3],$$

where $i^2 = -1$. The group contains a set of cyclic groups of period 2 on $x_1 = 0$. One such group is generated by $[x_1, -ix_2, ix_3]$. It is of period 4 on $x_2 = 0$ and $x_3 = 0$. The group in the plane is of order 120, as it contains the invariant reflection $[x_1, -x_2, -x_3]$ (Theorem 1).

Since we assume that no point or line remains invariant, there will be groups of elations whose centers do not lie on $x_1 = 0$ and whose axes do not pass through (100). We consider one such group. It must generate together with the reflection with center (100) and axis $x_1 = 0$ a group on the line joining their centers which does not contain an additive group of higher order than p^k. If $p^k > 3$, this group on the line must be of order $(p^k + 1)p^k(p^k - 1)/2$ or $(p^k + 1)p^k(p^k - 1)$. If $p^k = 3$, it must be of order 12, 24, or 60. In any case there will be a reflection having this line for axis, and having for center the point of intersection of the axis of the generating group of elations with $x_1 = 0$ (Theorem 1). The involution on $x_1 = 0$ performed by this reflection must leave invariant the group of the points on $x_1 = 0$.

In the special case where the group of the points on $x_1 = 0$ is of order $60\,(p^k = 3)$, the involution performed by this reflection on $x_1 = 0$ must belong to the group of order 60. Suppose for example that the center of the reflection is (010) and its axis $x_2 = 0$. The group of the points on $x_2 = 0$ contains a C_4 generated by $[x_1, -ix_2, ix_3]$. It must then be of order 24. But the fixed points of this C_4 will be interchanged by the group. The transformed transformation is $[ix_1, -ix_2, x_3]$. This is of period 4 on $x_1 = 0$. The group of the points on $x_1 = 0$ is now no longer of order 60 and hence it must contain an additive group of higher order than 3. We therefore exclude this case from the discussion entirely.

In the general case, in order that the reflection whose center lies on $x_1 = 0$ and whose axis passes through (100) shall permute among themselves the $p^k + 1$ centers of elations on $x_1 = 0$, either (i) its center must lie at some point conju-

gate with $(01I)$ and its axis pass through the conjugate point with respect to the $GF(p^k)$, or (ii) its center will coincide with one of the centers of elations and its axis will pass through one of the other centers.

We consider the case (i). Consider in particular the reflection with center $(01I)$ and axis $Ix_2 + x_3 = 0$. The axes of the elations with centers on $Ix_2 + x_3 = 0$ must all pass through $(01I)$. The group of the points on $Ix_2 + x_3 = 0$ contains a cyclic group of order $p^k + 1$ with fixed points (100) and $(01{-}I)$. It must then be of order $(p^k + 1)p^k(p^k - 1)$. Since there must be transformations leaving fixed $(01I)$ and interchanging (100) and $(01{-}I)$, the group of the points on $x_1 = 0$ must be of the same order. The axes of all elations whose centers lie on $x_1 = 0$ must then pass through (100). Hence a unique correspondence must be established between the centers and axes of elations and between the centers and axes of reflections. There can then be no centers of elations on any of the $p^k + 1$ axes of elations through (100) except those on $x_1 = 0$.

All the centers of elations except those on $x_1 = 0$ must then lie on the $p^k(p^k - 1)$ lines through (100) whose coördinates are in the $GF(p^{2k})$, but not in the $GF(p^k)$. Since there will be $p^k + 1$ on each of these lines, the total number of centers of elations will be

$$p^k + 1 + (p^k + 1)p^k(p^k - 1) = p^{3k} + 1.$$

All the centers not on $x_1 = 0$ will be conjugate under the group generated by the two groups of elations (E). We choose a center on $Ix_2 + x_3 = 0$ arbritrarily as $(-11{-}I)$. The $p^{3k} + 1$ centers will then be the points $(\epsilon, \alpha I + \beta, \gamma I + \delta)$, where $\epsilon, \alpha, \beta, \gamma, \delta$ are in the $GF(p^k)$ and $\alpha\delta - \beta\gamma = \epsilon^2$. (Fig. 8

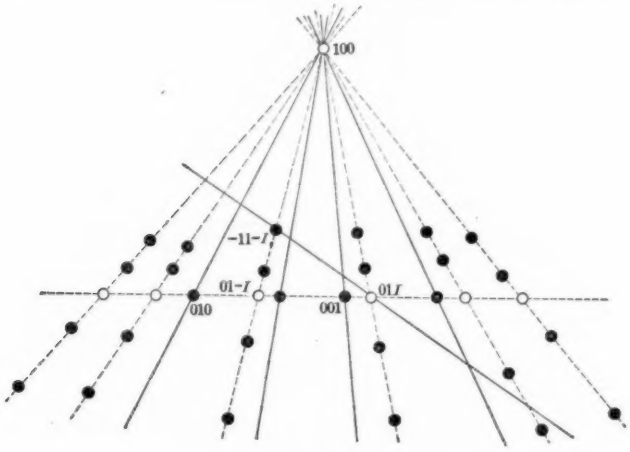

FIG. 8.

represents the configuration for $p^k = 3$.) The coefficients of the transformations permuting these points will lie in the $GF(p^{2k})$. We first seek to determine those groups which contain only transformations having for determinant a cube in the $GF(p^{2k})$.

The group will be transitive on the $p^{2k}(p^{2k} - p^k + 1)$ points whose coördinates lie in the $GF(p^{2k})$, but which are not centers of elations. The group of the points on any one of the axes of elations is metacyclic, containing p^{2k} involutions with a common fixed point. (The p^{3k} other axes of elations must meet that axis in p^{2k} points in sets of p^k each.) Hence (100) is conjugate with all the points on the axes of elations through it whose coördinates lie in the $GF(p^{2k})$ with the exception of the $p^k + 1$ centers of elations on $x_1 = 0$. It is also conjugate with all the points on the axes of reflections through it whose coördinates lie in the $GF(p^{2k})$, but which are not centers of elations.

Since all six permutations are made on the vertices of the triangle $(100)(01I)(01{-}I)$, there will be an homology of period $p^k + 1$ or $(p^k + 1)/2$ with center at each vertex (Theorem 3). But if $p^k + 1$ is divisible by 3, the determinant of an homology of period $p^k + 1$ will not be a cube in the $GF(p^{2k})$. Hence the period of the homology will be $(p^k + 1)/\nu$. (See Introduction.) The group leaving invariant (100) and $x_1 = 0$ is then of order

$$\frac{1}{\nu}(p^k+1)^2 p^k(p^k-1).$$

The whole group is of order

$$\frac{1}{\nu}(p^{2k} - p^k + 1)(p^k + 1)^2 p^{3k}(p^k - 1).$$

In order that the group of elations with center $(-11{-}I)$ shall permute among themselves the $p^{3k} + 1$ centers, we find that it must be given by

$$x_1' = \tfrac{1}{2}(2 + I\lambda)x_1 + \tfrac{1}{4}I\lambda x_2 - \tfrac{1}{4}\lambda x_3,$$

$$x_2' = -\tfrac{1}{2}I\lambda x_1 + \tfrac{1}{4}(4 - I\lambda)x_2 + \tfrac{1}{4}\lambda x_3,$$

$$x_3' = \tfrac{1}{2}I^2\lambda x_1 + \tfrac{1}{4}I^2\lambda x_2 + \tfrac{1}{4}(4 - I\lambda)x_3,$$

where λ takes all values in the $GF(p^k)$.

We may identify the group determined with the hyperorthogonal group,* $HO(3, p^{2k})$, by observing that it leaves invariant the function

$$x_1^{p^k+1} + \tfrac{1}{4}\left(x_2 + \frac{1}{I}x_3\right)^{p^k+1} - \tfrac{1}{4}\left(x_2 - \frac{1}{I}x_3\right)^{p^k+1}.$$

*Cf. L. E. DICKSON, Mathematische Annalen, vol. 52 (1899), pp. 561-581; also *Linear Groups*, Chap. V.

The locus of the points which are such that their coördinates satisfy this function equated to zero are the $p^{3k}+1$ centers of elations.

If p^k+1 is divisible by 3, a group exists containing the $HO(3, p^{2k})$ as a self-conjugate subgroup of index 3. It may be generated by the $HO(3, p^{2k})$ together with an homology of period p^k+1.

As follows from the unique choice of coördinates, there is a single conjugate set of $HO(3, p^{2k})$ under the whole collineation group.

We turn our attention again to the two groups of elations (E), and consider case (ii). If there is a group of elations with center on one of the axes through (100), and axis passing through the intersection of another one of those axes with $x_1=0$, the group on that axis must be of order $(p^k+1)p^k(p^k-1)$, since it contains a cyclic group of order p^k-1. Hence (100) will be conjugate with each of the p^k+1 centers of elations on $x_1=0$, since the fixed points of this cyclic group will be interchanged. Hence there will be at least p^k+1 axes of elations passing through each center. There can then be no centers not on any one of the p^k+1 axes through (100), since in that case we found that a polar configuration was determined. There will be p^k+1 centers on each of

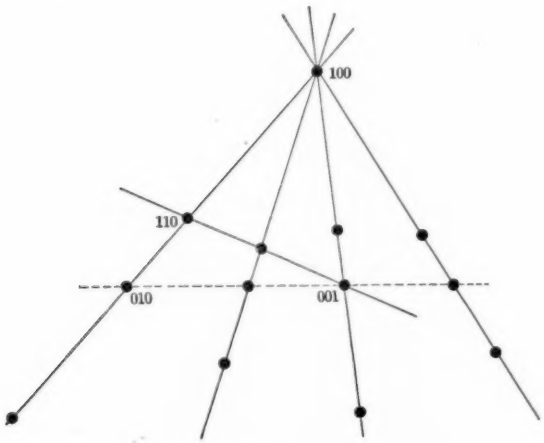

FIG. 9.

those p^k+1 axes, i. e., $p^{2k}+p^k+1$ in all. (Fig. 9 represents the configuration for $p^k=3$.)

If one of the centers on $x_3=0$ be (110), the $p^{2k}+p^k+1$ centers will be the points $(\alpha\beta\gamma)$, where α, β, γ lie in the $GF(p^k)$. The coefficients of the transformations in the group will then lie in the $GF(p^k)$. We consider first the groups containing only transformations the determinants of which are cubes in the $GF(p^k)$.

A group of elations with center (110) and axis passing through (001) will be given by

$$[(1+\lambda)x_1 - \lambda x_2,\ \lambda x_1 + (1-\lambda)x_2,\ x_3],$$

where λ takes all values in the $GF(p^k)$. Under the group generated by this group of elations together with the reflection with center (100) and axis $x_1 = 0$, (001) will remain invariant and there will be transformations interchanging (100) and (010). All six permutations will then be made on the vertices of the triangle (100)(010)(001). Hence there will be an homology of period $p^k - 1$ or $(p^k - 1)/2$ with center at each of the vertices (Theorem 3). But since if $p^k - 1$ is divisible by 3 the determinant of an homology of that period will not be a cube in the $GF(p^k)$, the period of the homology will be $(p^k - 1)/\mu$. (See Introduction.) The group of the points on any axis will be of order $(p^k + 1)p^k(p^k - 1)$, and consequently the group leaving (100) and $x_1 = 0$ fixed will be of order $(p^k + 1)p^k(p^k - 1)^2/\mu$. The group leaving fixed only (100) will be of order $(p^k + 1)p^{3k}(p^k - 1)^2/\mu$, and the whole group of order $(p^{2k} + p^k + 1)(p^k + 1)p^{3k}(p^k - 1)^2/\mu$. This is the $LF(3, p^k)$.*

If $p^k - 1$ is divisible by 3, a group exists containing the $LF(3, p^k)$ as a self-conjugate subgroup of index 3. It may be generated by the $LF(3, p^k)$ together with an homology of period $p^k - 1$.

§ 15. Subgroups of the Simple Groups $LF(3, p^k)$.

The subgroups of the $LF(3, p^k)$ are those groups which have been determined in this paper such that the coefficients of the transformations which they contain are in the $GF(p^k)$, and the determinant of any one of those transformations is a cube in that field, together with those subgroups which leave invariant a point, line, or triangle. There is a single conjugate set of each one of the groups under the whole collineation group of the plane. We suppose that the plane in which we are working is the $PG(2, p^k)$. If then the $LF(3, p^k)$ is the whole collineation group of the $PG(2, p^k)$, i. e., if $p^k - 1$ is not divisible by 3, there will be a single conjugate set of each subgroup; if $p^k - 1$ is divisible by 3, but a subgroup is invariant under a group of three times its order which is not a subgroup of the $LF(3, p^k)$, there will be also a single conjugate set of those subgroups; if however $p^k - 1$ is divisible by 3 and a subgroup is not invariant under a group which is not a subgroup of the $LF(3, p^k)$, then there will be three conjugate sets of those subgroups.

The subgroups are as follows:

1. Groups of order $(p^k + 1)p^{3k}(p^k - 1)^2/\mu$. Each of these groups leaves invariant a point and is isomorphic with a group of order $(p^k + 1)p^k(p^k - 1)$, permuting the lines through that point.

* Cf. L. E. Dickson, *Linear Groups*, pp. 75–78.

2. Groups of order $(p^k+1)p^{3k}(p^k-1)^2/\mu$. Each of these groups leaves invariant a line and is isomorphic with a group of order $(p^k+1)p^k(p^k-1)$, permuting the points on that line.

3. Groups of order $6(p^k-1)^2/\mu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(p^k)$, and makes all six permutations on its vertices.

4. Groups of order $3(p^{2k}+p^k+1)/\mu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(p^{3k})$, but not in the $GF(p^k)$, and permutes its vertices cyclically.

5. Groups of order $(p^k+1)p^k(p^k-1)$. Each such group leaves invariant a conic.

6. Groups of the same structure as that of the $LF(3, p^k)$ itself, i. e., the $LF(3, p^m)$, where m is a factor of k.

7. Groups containing the $LF(3, p^m)$ as self-conjugate subgroups of index 3 if p^m-1 is divisible by 3, and k/m is divisible by 3.

8. The hyperorthogonal groups, $HO(3, p^{2m})$, where $2m$ is a factor of k.

9. Groups containing the $HO(3, p^{2m})$ as self-conjugate subgroups of index 3 if p^m+1 is divisible by 3, and $k/2m$ is divisible by 3.

10. The Hessian groups of order 216 (if p^k-1 is divisible by 9), 72 and 36 (if p^k-1 is divisible by 3).

11. Groups of order 168, which exist if $\sqrt{-7}$ exists in the $GF(p^k)$, i. e., if k is even, or (by the law of quadratic reciprocity) if k is odd and $p=7$, $7f+1$, $7f+2$, or $7f+4$. For $p=7$ any one of these groups leaves invariant a conic.

12. Groups of order 360, which exist if both $\sqrt{5}$ and a cube root of unity exist in the $GF(p^k)$; i. e., if k is even, or if k is odd, provided $p=15f+1$ or $15f+4$. For $p=3$ any one of these groups leaves invariant a conic.

13. Groups of order 720 containing the groups of order 360 as self-conjugate subgroups. These exist only for $p=5$ and k even.

14. Groups of order 2520, each isomorphic with the alternating group on seven letters. These exist only for $p=5$ and k even.

Since the largest subgroup is of order $(p^k+1)p^{3k}(p^k-1)^2/\mu$, we have

Theorem 29. *The smallest number of letters on which the group, $LF(3, p^k)$, may be represented as a permutation group is $p^{2k}+p^k+1$.*

§ 16. Subgroups of the Simple Groups $HO(3, p^{2k})$.

The subgroups of the $HO(3, p^{2k})$ are those groups determined above which have an invariant conjugate with

$$x_1^{p^k+1}+x_2^{p^k+1}+x_3^{p^k+1}=0,$$

together with the subgroups which leave invariant a point, line, or triangle. Each set of subgroups forms a single conjugate set provided the $HO(3, p^{2k})$ is invariant under no larger group under the whole collineation group of the plane, i. e., if $p^k + 1$ is not divisible by 3. If $p^k + 1$ is divisible by 3, each set of subgroups will form a single conjugate set if any one of them is invariant under a group of three times its order which is a subgroup of the group under which the $HO(3, p^{2k})$ is invariant but not of the $HO(3, p^{2k})$ itself; otherwise there will be three conjugate sets.

The subgroups are as follows:

1. Groups of order $(p^k + 1)p^{3k}(p^k - 1)/\nu$. Any such group leaves invariant the center and axis of a group of elations and is isomorphic with a metacyclic group on the axis of order $(p^{2k} - 1)p^{2k}/\nu$.

2. Groups of order $(p^k + 1)^2 p^k (p^k - 1)/\nu$. Any such group leaves invariant the center and axis of an homology and is isomorphic with the group of the points on the axis, which is of order $(p^k + 1)p^k(p^k - 1)$.

3. Groups of order $6(p^k + 1)^2/\nu$. Any such group leaves invariant a triangle whose coördinates are in the $GF(p^{2k})$ and makes all six permutations on its vertices.

4. Groups of order $3(p^{2k} - p^k + 1)/\nu$. Any such group leaves invariant a triangle whose coördinates lie in the $GF(p^{6k})$, but not in the $GF(p^{2k})$, and permutes its vertices cyclically.

5. Groups of order $(p^k + 1)p^k(p^k - 1)$. Any such group leaves invariant a conic.

6. The hyperorthogonal groups, $HO(3, p^{2m})$, where m is a factor of k and k/m is odd.

7. Groups containing the $HO(3, p^{2m})$ as self-conjugate subgroups of index 3 if $p^m + 1$ is divisible by 3 and k/m is odd and is divisible by 3.

8. The Hessian groups of order 216 (if $p^k + 1$ is divisible by 9), 72 and 36 (if $p^k + 1$ is divisible by 3).

9. Groups of order 168, which exist if $\sqrt{-7}$ does not exist in $GF(p^k)$, i. e., if k is odd and $p = 7f + 3$, $7f + 5$, or $7f + 6$. For $p = 7$ these groups also appear, but each leaves invariant a conic.

10. Groups of order 360, which exist if $\sqrt{5}$ exists and a cube root of unity does not exist in the $GF(p^k)$, i. e., if k is odd and $p = 5$, $15f - 1$, or $15f - 4$. If k is even and $p = 3$ these groups also appear, but in that case each leaves invariant a conic.

11. Groups of order 720, which exist if $p = 5$ and k is odd.

12. Groups of order 2520, which exist if $p = 5$ and k is odd.

The largest subgroup of the $HO(3, p^{2k})$ is of order $(p^k + 1)p^{3k}(p^k - 1)/\nu$, except for $p^k = 5$, in which case the largest subgroup is of order 2520. Hence we have

Theorem 30. *The smallest number of letters on which the group* $HO(3, p^{2k})$ *may be represented as a permutation group is* $p^{3k}+1$, *except for* $p^k = 5$, *in which case the smallest number is* 50.

That the $HO(3, p^{2k})$ can be represented as a permutation group on $p^{3k}+1$ letters has been shown by Dickson (Mathematische Annalen, vol. 55, p. 532). Similar results are obtained by him in that paper for the general hyperorthogonal group in m variables.

NEW HAVEN, CONN.,
September, 1910.

GENERAL THEORY OF LINEAR DIFFERENCE EQUATIONS*

BY

GEORGE D. BIRKHOFF

INTRODUCTION.

The theory of linear difference equations with rational coefficients was in a very backward state until POINCARÉ † in 1882 developed the notion of asymptotic representation, and its application to this branch of mathematics. Further important progress in this direction has been made only very recently, notably by GALBRUN, ‡ who by means of the Laplace transformation has investigated the properties of a set of fundamental solutions in the entire plane of the complex variable x.

The aim of the present paper is first to study the nature of these solutions to which I have been led by means of direct methods, and secondly to show that there exists a purely Riemannian theory of linear difference equations. In particular certain rational functions of $e^{2\pi\sqrt{-1}\,x}$ are shown to play a part like that of the monodromic group constants of an ordinary linear differential equation.

To the best of my knowledge, the importance of the functional standpoint in the field of difference equations was emphasized first by VAN VLECK in an inspiring series of lectures given at the University of Wisconsin in the spring of 1909, in which he conjectured the existence of sets of solutions analytic on either the left or the right side of the complex plane.

On account of the extreme simplicity of the matrix notation, I have found it convenient to deal with a linear difference system of n equations of the first order

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x) g_j(x), \qquad (i = 1, 2, \cdots, n), \tag{1}$$

rather than with a single equation of the nth order. In these equations the functions $a_{ij}(x)$ are taken to be rational functions of x with a pole at $x = \infty$ of order μ at most, so that one has

$$a_{ij}(x) = a_{ij} x^{\mu} + a_{ij}^{(1)} x^{\mu-1} + \cdots, \qquad (|x| > R). \tag{2}$$

* Presented to the Society, September 6, 1910.

† American Journal of Mathematics, vol. 7 (1885), pp. 203–258.

‡ Comptes Rendus, vol. 148 (1909), pp. 905–907.

The sole other restriction to be imposed on these functions below will, in particular, be satisfied if the determinantal *characteristic equation*

$$|a_{ij} - \delta_{ij}\rho_j| = 0 \tag{3}$$

[where $\delta_{ij} = 0$, $i \neq j$; $\delta_{ii} = 1$] has n distinct roots which are not zero. If the n sets of functions

$$\begin{array}{c} g_{11}(x), \cdots, g_{n1}(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g_{1n}(x), \cdots, g_{nn}(x) \end{array}$$

are n linearly independent solutions of (1), the array of elements $g_{ij}(x)$ forms a *matrix solution* $G(x)$. The array of elements $a_{ij}(x)$ forms a second matrix $A(x)$, and the n^2 equations which the n solutions satisfy can be combined into a single matrix equation

$$G(x+1) = A(x)G(x). \tag{4}$$

If x be replaced by $x-1$, this may be written

$$G(x-1) = A^{-1}(x-1)G(x). \tag{4}'$$

The most general matrix solution of (4) is furnished by

$$H(x) = G(x)P(x), \tag{5}$$

in which $G(x)$ is any particular matrix solution, and $P(x)$ is any arbitrary matrix of periodic functions of period 1 whose determinant does not vanish identically.

The equation (4) admits of *two symbolic solutions*

$$\begin{cases} G(x) = A(x-1)A(x-2)\cdots, \\ G(x) = A^{-1}(x)A^{-1}(x+1)\cdots, \end{cases}$$

which converge to limit matrices only in particular cases. This convergence is secured to a sufficient extent by a suitable modification, made in § 1, as follows: The system (1) is known to be satisfied formally by taking for $g_1(x)$, $g_2(x)$, $\cdots, g_n(x)$ certain formal series, one set of them for each of the n roots ρ of the characteristic equation (3). The matrix formed from these sets taken in a certain order is denoted by $S(x)$, and the modified matrix obtained by breaking off these elements at the kth term is called $T(x)$. The sequences of matrices

$$\begin{array}{l} A(x-1)A(x-2)\cdots A(x-m)T(x-m), \\ A^{-1}(x)A^{-1}(x+1)\cdots A^{-1}(x+m-1)T(x+m), \end{array} \quad (m=1, 2, \cdots),$$

form the desired modification of the two symbolic solutions. It is demonstrated that all the determinants formed from the first λ columns and any λ rows of the

first of these matrices, ($\lambda = 1, 2, \cdots, n$), or from the last λ columns and any λ rows of the second of these matrices, converge to limit functions independent of k as m becomes infinite. By the aid of these functions and by a process of summation based on a contour integral suggested by the one which CARMICHAEL* has used, two remarkable particular matrix solutions $G(x)$ and $H(x)$ are obtained in §§ 2, 3, 4. These are termed the *first and second principal matrix solutions* respectively. The elements of $G(x)$ are analytic, save for poles at points $\sigma + 1, \sigma + 2, \cdots$, where σ is a pole of one of the elements of $A(x)$, and have the asymptotic form of the corresponding elements of $S(x)$ in any left half plane. The elements of $H(x)$ are analytic save for poles at points $\sigma - 1, \sigma - 2, \cdots$, where σ is a pole of one of the elements of $A^{-1}(x-1)$, and have the asymptotic form of the corresponding elements of $S(x)$ in any right half plane. The principal matrix solutions $G(x)$ and $H(x)$ are uniquely determined by this asymptotic property. A relation (5) will also, of course, subsist between $G(x)$ and $H(x)$.

Conversely if $G(x)$ and $H(x)$ are asymptotically represented by the elements of a matrix $S(x)$ *even to terms of the first order* and are related by means of a matrix $P(x)$ of periodic functions, as in (5), these matrices $G(x)$ and $H(x)$ will form the first and second principal matrix solutions of a system (4) in which the elements of $A(x)$ are rational (see § 7).

It is proved in § 5 that the elements of $P(x)$ are rational functions of $e^{2\pi\sqrt{-1}x}$. The structure of these functions is determined with particular reference to the case when the functions $a_{ij}(x)$ are polynomials of degree μ, to which case the more general one is reduced.

These results lead in § 6 to the determination of the nature of the elements of $G(x)$ and $H(x)$ in the remaining right and left half plane respectively; the asymptotic form is found to alter along certain *critical rays*. A complete description of the principal matrix solutions is thus obtained.

In § 7 it is shown that the number of *characteristic constants* involved in the complete characterization of $G(x)$ and $H(x)$ and their connection by means of $P(x)$ is precisely equal to the number of arbitrary constants in the coefficients $a_{ij}(x)$ assumed to be polynomials, and that the characteristic constants are independent. Thus I have ventured to propose the following as a fundamental problem of linear difference equations: *to determine whether there exists a linear difference system with prescribed characteristic constants.*

It would be interesting to know the exact relation between the characteristic constants and the monodromic group constants of the differential equation yielded by the Laplace transformation.

The method contained in the present paper is of wide generality and applies if the coefficients $a_{ij}(x)$ have the character of rational functions only at $x = \infty$,

*These Transactions, vol. 12 (1911), pp. 99–134.

in all cases where there exist n formal solutions. CARMICHAEL (loc. cit.) has studied this case to a considerable extent; in particular he has proved the existence of analytic solutions and has partly discussed their asymptotic form in the direction of the axis of reals.

§ 1. THE DETERMINANT LIMITS.

It is apparent that the given linear system (1), when written in the matrix notation (4), (4)′, possesses the two symbolic solutions

$$A(x-1)A(x-2)\cdots, \qquad A^{-1}(x)A^{-1}(x+1)\cdots.$$

These will converge to limit matrices in certain special cases, for example if $a_{ij}(x)$ has the form

$$\delta_{ij} + \frac{a_{ij}^{(2)}}{x^2} + \frac{a_{ij}^{(3)}}{x^3} + \cdots;$$

in this case the above symbolic solutions will yield the complete solution of (1). In general however these expressions must be properly modified in order to be made to lead indirectly to a complete solution of (1).

Let us first make precise the limitation which is to be made on the coefficients $a_{ij}(x)$ in equations (1). If the roots of the characteristic equation (3) are distinct, and if none of them are zero, there will exist precisely n sets of series

$$(6)\qquad \begin{aligned} s_{1j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{1j} + \frac{s_{1j}^{(1)}}{x} + \cdots \right\}, \\ s_{2j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{2j} + \frac{s_{2j}^{(1)}}{x} + \cdots \right\}, \quad (j=1, 2, \cdots, n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{nj}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{ s_{nj} + \frac{s_{nj}^{(1)}}{x} + \cdots \right\}, \end{aligned}$$

each of which constitutes a formal solution of (1). The proof of this is a matter of direct reckoning, and the fact will be taken for granted.* The constants $\rho_1, \rho_2, \cdots, \rho_n$ are the roots of the characteristic equation, and furthermore we have the determinant

$$(7)\qquad d = |s_{ij}| \neq 0.$$

* The formal solutions of a single linear difference equation of the nth order have been frequently used (for example see HORN, Crelle's Journal, vol. 138 (1910), pp. 159–191) and admit of easy extension to a linear system. The formal solutions may be obtained by direct substitution if the roots of the characteristic equation are distinct and different from zero; the series expansion

$$\left(1+\frac{1}{x}\right)^{\mu x} = e^{\mu x \log\left(1+\frac{1}{x}\right)} = e^{\mu} - \frac{\mu e^{\mu}}{2x} + \cdots$$

enables one to make a comparison of coefficients and thus to determine the constants of the formal solutions.

The restrictive hypothesis which we shall make is not that $\rho_1, \rho_2, \cdots, \rho_n$ *are distinct but only that there exist* n *formal solutions* (6) *with the specified property* (7). The quantities $\rho_1, \rho_2, \cdots, \rho_n$ must of course still be the roots of the characteristic equation, and none of them can be zero. We shall choose our subscripts so that

$$|\rho_1| \geqq |\rho_2| \geqq \cdots \geqq |\rho_n|. \tag{8}$$

Without loss of real generality we may add the further condition that the linear homogeneous difference equation of the nth order at most which $g_1(x)$ satisfies, and which is obtained from (1) by elimination, is actually of the nth order.

The elements $s_{ij}(x)$ form a matrix $S(x)$ which is a formal matrix solution of the equation (4)

$$S(x+1) = A(x)\, S(x).$$

Let us compute the elements $s'_{ij}(x)$ of the inverse matrix $S^{-1}(x)$. The determinant $|S(x)|$ may be written

$$x^{n\mu x}(\rho_1\rho_2 \cdots \rho_n e^{-n\mu})^x x^{r_1+r_2+\cdots+r_n}\left\{d + \frac{d^{(1)}}{x} + \cdots\right\}.$$

The element in the ith row and jth column of the inverse matrix is the quotient of the cofactor of the element in the jth row and the ith column of this determinant by the determinant itself, and is therefore given by a formal series

$$s'_{ij}(x) = x^{-\mu x}(\rho_i e^{-\mu})^{-x} x^{-r_i}\left\{\sigma_{ij} + \frac{\sigma^{(1)}_{ij}}{x} + \cdots\right\}. \tag{9}$$

Let $T(x)$ denote any matrix obtained by breaking off the elements of $S(x)$ at the kth term, or more generally by replacing $s_{ij}(x)$ by convergent series $t_{ij}(x)$ of the same form as $s_{ij}(x)$ and having the same first k terms. Write then

$$T(x+1) = B(x)\, T(x). \tag{10}$$

The matrix $B(x)$, as thus defined, is clearly a matrix of functions $b_{ij}(x)$ each of which has an expansion like $a_{ij}(x)$ at $x = \infty$ with precisely the same first k terms. For if we compare the formulas

$$B(x) = T(x+1)\, T^{-1}(x) \qquad \text{and} \qquad A(x) = S(x+1)\, S^{-1}(x),$$

we observe first that the element $t'_{ij}(x)$ of $T^{-1}(x)$ is given by (9) to the first k terms, and that in consequence the right hand members of both the equations have the same first k terms. In consequence we may put

$$A(x) = B(x)\,[1 + x^{-k}\, C(x)], \tag{11}$$

1 being the identity matrix, and the elements of $C(x)$ being analytic at $x = \infty$. It is to be noted that the determinant formed from the principal coefficients of

$A(x)$ and $B(x)$, namely $|a_{ij}|$, cannot be zero, for that implies that zero is a root of the characteristic equation.

In this paper it is necessary to distinguish sharply between two kinds of asymptotic representation of a function by a series of the form (6), say

$$s(x) = x^{\mu x}(\rho e^{-\mu})^x x^r\left(s + \frac{s^{(1)}}{x} + \cdots\right).$$

Let $g(x)$ be the given function; if for each k the difference

$$g(x)x^{-\mu x}(\rho e^{-\mu})^{-x}x^{-r} - \left(s + \frac{s^{(1)}}{x} + \cdots + \frac{s^{(k)}}{x^k}\right)$$

becomes uniformly small of order x^{-k} for $\lim x = \infty$ in a certain region, we say that $g(x)$ is asymptotically represented by $s(x)$ in that region, *with respect to* x; if on the other hand the difference is uniformly small of order v^{-k} (where $x = u + \sqrt{-1}\,v$, u, v being real quantities) for $\lim v = \infty$ in the region, we say that the representation is *with respect to* v. In this connection two important inequalities are

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{(x-\nu)^k}\right| < \frac{\pi}{2\,|x|^{k-1}} \qquad (k \geqq 2), \tag{12}$$

for x in the left half of the complex plane, and

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{(x-\nu)^k}\right| < \frac{2}{|v|^{k-1}}\left[\frac{1}{|v|} + \frac{\pi}{2}\right] \qquad (k \geqq 2), \tag{13}$$

for x in the right half of the complex plane. To prove the first of these we note that when x lies in the left half plane u is negative, so that

$$|x-\nu|^2 = (u-\nu)^2 + v^2 > u^2 + v^2 + \nu^2,$$

whence the sum of the given series is less than

$$\sum_{\nu=1}^{\infty}\frac{1}{(|x|^2+\nu^2)^{\frac{1}{2}k}} < \int_0^{\infty}\frac{1}{|x|^{k-2}}\,\frac{d\nu}{(|x|^2+\nu^2)} = \frac{\pi}{2\,|x|^{k-1}}.$$

Again, for x in the right half plane, our series may be written

$$\sum_{\nu=1}^{\infty}\frac{1}{[(u-\nu)^2+v^2]^{\frac{1}{2}k}} < \sum_{\nu=-\infty}^{+\infty}\frac{1}{[(u-\nu)^2+v^2]^{\frac{1}{2}k}} < \frac{1}{|v|^{k-2}}\sum_{\nu=-\infty}^{+\infty}\frac{1}{(u-\nu)^2+v^2}.$$

If each quantity $|u-\nu|$ be replaced by the last of the integers $0, 1, 2, \cdots$ which does not exceed it, the terms in the last series are not decreased, and hence one has

$$\frac{1}{|v|^{k-2}}\sum_{\nu=-\infty}^{+\infty}\frac{1}{(u-\nu)^2+v^2} \leqq \frac{2}{|v|^{k-2}}\sum_{\nu=0}^{\infty}\frac{1}{\nu^2+v^2} < \frac{2}{|v|^{k-2}}\left[\frac{1}{v^2} + \int_0^{\infty}\frac{d\nu}{\nu^2+v^2}\right].$$

But the last integral reduces to $\pi/2|v|$, and thus the second inequality holds also.

We may now proceed to the existence theorem:

THEOREM I. *Form the two sequences of matrices*

$$(14)\qquad \begin{aligned} P_m(x) &= A(x-1)A(x-2)\cdots A(x-m)T(x-m),\\ Q_m(x) &= A^{-1}(x)A^{-1}(x+1)\cdots A^{-1}(x+m-1)T(x+m), \end{aligned} \qquad (m=1,2,\cdots).$$

Every determinant formed from the first λ *columns* $(\lambda=1,2,\cdots,n)$ *and the ith, jth, ···, lth rows of* $P_m(x)[Q_m(x)]$ *converges, for* k *sufficiently large, to a definite limit function* $u_{ij\cdots l}(x)[v_{ij\cdots l}(x)]$, *independent of* k, *as* m *becomes infinite. This function is analytic throughout the entire* x*-plane excepting at points any number of units to the right (left) of poles of the elements of* $A(x)[A^{-1}(x-1)]$ *and at these points may have a pole. The asymptotic form of* $u_{ij\cdots l}(x)[v_{ij\cdots l}(x)]$ *in the vicinity of* $x=\infty$ *is given by the corresponding determinant* $s_{ij\cdots l}(x)$ *of* $S(x)$, *with respect to* x *in any left (right) half of the plane, and with respect to* v *in any right (left) half of the plane.*

Proof. We shall first prove the theorem in so far as it relates to the case $\lambda=1$, and then show how the cases $\lambda=2,3,\cdots,n$ may be reduced to this case.

It is necessary first to show that the elements of the first column of $P_m(x)$ converge to definite limit functions $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$ as m becomes infinite.

The matrix $P_m(x)$ may be written

$$\begin{aligned} P_m(x) &= T(x)\bar{P}_m(x)\\ &= T(x)\{[T^{-1}(x)A(x-1)T(x-1)][T^{-1}(x-1)A(x-2)T(x-2)]\\ &\qquad \cdots [T^{-1}(x-m+1)A(x-m)T(x-m)]\}. \end{aligned}$$

It is sufficient to prove that the elements of the first column of $\bar{P}_m(x)$ will converge, inasmuch as the elements of the first column of $P_m(x)$ arise from combination of the rows of $T(x)$ with the first column of $\bar{P}_m(x)$. The matrix $\bar{P}_m(x)$ is a product of matrices, the type of each of which is

$$T^{-1}(x+1)A(x)T(x).$$

By (10) and (11) this reduces to

$$1+\frac{1}{x^k}T^{-1}(x)C(x)T(x).$$

Let us examine the second term. The element in the ith row and jth column of $T^{-1}(x)$ will be given by an expression like that for $s'_{ij}(x)$ in (9) with the same first k terms; the elements of $C(x)$ are analytic at $x=\infty$, the element

in the ith row and jth column of $T(x)$ will be given by an expression like that for $s_{ij}(x)$ and with the same first k terms. The element in the ith row and jth column of the product $T^{-1}(x)\,C(x)\,T(x)$ has therefore the form

$$\left(\frac{\rho_j}{\rho_i}\right)^x x^{r_j-r_i} \text{ times a function analytic at } x=\infty.$$

Choose first k so large that it exceeds every difference $r_j - r_i$ in absolute value by more than an assigned integer $d \geqq 2$, and we see that the second term may be written

$$\frac{1}{x^d}\Theta(x) \quad \text{with elements} \quad \theta_{ij}(x) = \left(\frac{\rho_j}{\rho_i}\right)^x \phi_{ij}(x). \tag{15}$$

The function $\phi_{ij}(x)$ is finite in the vicinity of $x=\infty$, at least if x be restricted not to pass the positive axis of reals.

By means of this notation, $\bar{P}_m(x)$ reduces to the form

$$\left(1+\frac{1}{(x-1)^d}\Theta(x-1)\right)\left(1+\frac{1}{(x-2)^d}\Theta(x-2)\right)\cdots\left(1+\frac{1}{(x-m)^d}\Theta(x-m)\right),$$

or more fully,

$$1+\sum_{k_1}\frac{1}{(x-k_1)^d}\Theta(x-k_1) + \sum_{k_1,k_2}^{k_1<k_2}\frac{1}{(x-k_1)^d(x-k_2)^d}\Theta(x-k_1)\Theta(x-k_2)+\cdots. \tag{16}$$

It is the elements of the first column of this matrix which must be shown to converge. According to the notation of (15), the ith element is

$$\delta_{i1}+\sum_{k_1}\frac{1}{(x-k_1)^d}\left(\frac{\rho_1}{\rho_i}\right)^{x-k_1}\phi_{i1}(x-k_1)+\sum_{k_1,k_2}^{k_1<k_2}\frac{1}{(x-k_1)^d(x-k_2)^d} \times\sum_{\tau=1}^{n}\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_1}{\rho_\tau}\right)^{x-k_2}\phi_{i\tau}(x-k_1)\phi_{\tau 1}(x-k_2)+\cdots. \tag{17}$$

But $\phi_{ij}(x)$ is limited in the vicinity of $x=\infty$ and is less than some constant M for $|x|>R_1$. Therefore if all the points $x-1, x-2, \cdots$ are exterior to the circle $|x|=R_1$, which is true if x lies in D of fig. 1, the typical element in

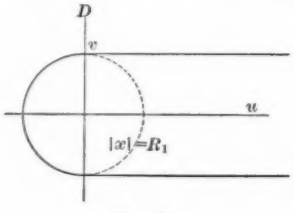

FIG. 1.

the $(l+1)$th term of (17),

$$\frac{1}{(x-k_1)^d}\frac{1}{(x-k_2)^d}\cdots\frac{1}{(x-k_l)^d}\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_\sigma}{\rho_\tau}\right)^{x-k_2}\cdots\left(\frac{\rho_1}{\rho_\nu}\right)^{x-k_l}$$
$$\times\,\phi_{i\tau}(x-k_1)\phi_{\tau\sigma}(x-k_2)\cdots\phi_{\nu 1}(x-k_l),$$

is less in absolute value than the quantity

$$\left(\frac{\rho_1}{\rho_i}\right)^x\frac{M^l}{(x-k_1)^d\cdots(x-k_l)^d}.$$

This inequality depends on the fact that we have

$$\left|\left(\frac{\rho_\tau}{\rho_i}\right)^{x-k_1}\left(\frac{\rho_\sigma}{\rho_\tau}\right)^{x-k_2}\cdots\left(\frac{\rho_1}{\rho_\nu}\right)^{x-k_l}\right|$$
$$=\left|\left(\frac{\rho_1}{\rho_i}\right)^x\left(\frac{\rho_i}{\rho_1}\right)^{k_1}\left(\frac{\rho_\tau}{\rho_1}\right)^{k_2-k_1}\cdots\left(\frac{\rho_\nu}{\rho_1}\right)^{k_l-k_{l-1}}\right|\leqq\left|\left(\frac{\rho_1}{\rho_i}\right)^x\right|$$

as a consequence of (8), and the fact that $k_1, k_2, \cdots, k_n$ are an increasing set of positive integers. Hence each term of (17) does not exceed the corresponding term of

$$\delta_{i1}+\left|\left(\frac{\rho_1}{\rho_i}\right)^x\right|\sum_{k_1}\frac{M}{|x-k_1|^d}+\left|\left(\frac{\rho_1}{\rho_i}\right)^x\right|\sum_{k_1,k_2}^{k_1<k_2}\frac{nM^2}{|x-k_1|^d|x-k_2|^d}+\cdots \tag{18}$$

in absolute value.

Allow m to approach infinity. The expression (17) becomes a multiple series. Since the sum of the terms after the first in (18) approaches the limit

$$\frac{1}{n}\left|\left(\frac{\rho_1}{\rho_i}\right)^x\right|\left\{\left[\left(1+\frac{nM}{|x-1|^d}\right)\left(1+\frac{nM}{|x-2|^d}\right)\cdots\right]-1\right\}, \tag{19}$$

it is obvious that the elements of the first column of $\bar{P}_m(x)$ and therefore of $P_m(x)$ converge absolutely and uniformly to an analytic function in the vicinity of points in D. Moreover as we may write

$$P_m(x)=A(x-1)A(x-2)\cdots A(x-r)P_{m-r}(x-r),$$

and as we may take r so large that $x-r-1, x-r-2, \cdots$ lie without the circle $|x|=R_1$, the elements of the first column of $P_m(x)$ will converge uniformly in the vicinity of any point of the plane excepting those only at which one of the elements of $A(x-1), A(x-2), \cdots$ has a pole. Hence the ith element of the first column of $P_m(x)$ converges to a function $u_i(x)$ which is analytic except at the points *congruent to* (any number of units to right or left of) the poles of the elements of $A(x)$ on the right. At these points $u_i(x)$ may have a pole.

It may easily be made evident that the definition of $u_1(x), u_2(x), \cdots, u_n(x)$

is independent of the choice of k and T. For if k' is any second choice of k and $T'(x)$ is the corresponding function $T(x)$, we may nevertheless write

$$P'_m(x) = T(x)\{[T^{-1}(x)A(x-1)T(x-1)]$$
$$\times [T^{-1}(x-1)A(x-2)T(x-2)]\cdots[T^{-1}(x-m+1)A(x-m)T'(x-m)].$$

If the elements of the first column be expanded in a sum like (17) and m be allowed to become infinite, the resulting multiple series for $u_i(x)$ will be term for term identical with that obtained before.

It remains to discuss the asymptotic nature of $u_1(x), \cdots, u_n(x)$ for large values of $|x|$. The limit $\bar{u}_i(x)$ of the ith element of the first column of $\bar{P}_m(x)$ differs from δ_{i1} by a quantity less than (19) in absolute value if x lies in D. This quantity is itself less than

$$\frac{1}{n}\left(\frac{\rho_1}{\rho_i}\right)^x\left(e^{\sum_{\nu=0}^{\infty}\frac{nM}{x-\nu}}-1\right)$$

in absolute value. If now any line be drawn parallel to the axis of imaginaries and if x be restricted to lie in D to the left of the line, by inequality (12) the last factor will be of the form $M(x)/x^{d-1}$, where $M(x)$ is limited. Thus we may write

$$\bar{u}_i(x) = \delta_{i1} + \left(\frac{\rho_1}{\rho_i}\right)^x \frac{M_i(x)}{x^{d-1}}, \quad |M_i(x)| < M$$

for x in D and to the left of this line. Likewise if x is in D to the right of such a line, by inequality (13) we may write

$$\bar{u}_i(x) = \delta_{i1} + \left(\frac{\rho_1}{\rho_i}\right)^x \frac{M'_i(x)}{x^{d-1}}, \quad |M'_i(x)| < M.$$

But the relation $P_m(x) = T(x)\bar{P}_m(x)$ gives at once

$$u_i(x) = \sum_{\tau=1}^{n} t_{i\tau}(x)\bar{u}_\tau(x).$$

If in this equation the first of the above expressions for $\bar{u}_1(x), \bar{u}_2(x), \cdots, \bar{u}_n(x)$ be substituted, we find at once

$$u_i(x) = t_{i1}(x) + \sum_{\tau=1}^{n} t_{i\tau}(x)\left(\frac{\rho_1}{\rho_\tau}\right)^x \frac{M_\tau(x)}{x^{d-1}}.$$

The functions $t_{ij}(x)$ are the same in form as $s_{ij}(x)$ up to the kth term. Substituting in their expansions we obtain

$$u_i(x) = x^{\mu_i x}\rho_1^x x^{r_1}\left\{\left(s_{i1} + \frac{s_{i1}^{(1)}}{x} + \cdots\right) + \sum_{\tau=1}^{n}\frac{x^{r_\tau}}{x^{r_1}}\frac{M_\tau(x)}{x^{d-1}}\left(s_{i\tau} + \frac{s_{i\tau}^{(1)}}{x} + \cdots\right)\right\}.$$

The second term in brackets can clearly be made infinitesimal in x of as high

order as may be desired by taking k, and therefore d, large enough. Consequently $u_i(x)$ is represented asymptotically by $s_{i1}(x)$ with respect to x to the left of any such line and likewise with respect to v to the right of such a line. The part of the theorem that relates to the case $\lambda = 1$ is proved.

It is clear that if the subscripts $1, 2, \cdots, n$ had not been so chosen that (8) was true, the only modification would be that the elements of any ith column of $P_m(x)$ for which the quantity ρ_i had a maximum absolute value would converge.

The case $\lambda = 2$ will next be discussed. This reduces at once to the case $\lambda = 1$ *inasmuch as the two-rowed determinants* $u^{(m)}_{\alpha\beta}(x)$ *formed from the elements of the first two columns of* $P^{(m)}(x)$ *which are in the* α*th and* β*th rows* $(\alpha < \beta)$ *are related to a certain difference system precisely as the elements of the first column are related to the original system* (1).

In fact, form the system with general solution

$$g_{ij:}(x) = \begin{vmatrix} g_i'(x) & g_i''(x) \\ g_j'(x) & g_j''(x) \end{vmatrix} \qquad (i, j = 1, 2, \cdots, n;\ i < j). \tag{20}$$

Here $g_1'(x), g_2'(x), \cdots, g_n'(x)$ and $g_1''(x), g_2''(x), \cdots, g_n''(x)$ are any pair of solutions of (1). One finds by the use of (1)

$$g_{ij:}(x+1) = \sum_{k,\,l} a_{ij:\,kl}(x)\, g_{kl:}(x), \tag{21}$$

where

$$a_{ij:\,kl}(x) = a_{ik}(x)\, a_{jl}(x) - a_{il}(x)\, a_{jk}(x).$$

Accordingly we have $\frac{1}{2}n(n-1)$ linear difference equations of precisely the type (1) to determine the functions $g_{ij:}(x)$. It is clear that the sets of formal expansions

$$s_{ij:\,kl}(x) = \begin{vmatrix} s_{ik}(x) & s_{il}(x) \\ s_{jk}(x) & s_{jl}(x) \end{vmatrix}$$

$$= x^{2\mu x}(\rho_k \rho_l)^x x^{r_k + r_l} \{ (s_{ik}\, s_{jl} - s_{il}\, s_{jk}) + \text{terms in } x^{-1}, x^{-2}, \cdots \},$$

k and l fixed, will satisfy (21). There are $\frac{1}{2}n(n-1)$ sets of these, and furthermore the determinant of the constant terms

$$|s_{ik}\, s_{jl} - s_{il}\, s_{ik}| \qquad (i < j;\ k < l)$$

is precisely equal to d^{n-1} and is therefore not zero by (7).* Hence the system (21) satisfies the restrictive hypothesis made at the beginning of § 1.

We are now in a position to apply the result already obtained. In matrix notation, equations (21) may be expressed as

$$G_2(x+1) = A_2(x) G_2(x).$$

* PASCAL: *Die Determinanten*, p. 87.

Also let $T_2(x)$ be the matrix which the formal matrix solution $S_2(x)$ becomes when the formal series are replaced by convergent series which agree with them to the kth term. *The elements in that column of*

$$P_{2m}(x) = A_2(x-1)A_2(x-2)\cdots A_2(x-m)T_2(x-m)$$

for which the quantity $\rho_k\rho_l$ in the formal solution which appears in the same column of $S_2(x)$ has a maximum absolute value will converge; this results from an application of the preceding result for $\lambda = 1$ to the system of $\frac{1}{2}n(n-1)$ equations. Equations (8) show that we may take $k=1$, $l=2$ and it is also clear that we may take the elements of this column of $T_2(x)$ to be

$$t_{ij:12}(x) = \begin{vmatrix} t_{i1}(x) & t_{i2}(x) \\ t_{j1}(x) & t_{j2}(x) \end{vmatrix}.$$

The elements $p^{(m)}_{ij:12}(x)$ of this column of $P_{2m}(x)$ converge to functions analytic everywhere except for poles at points to the right of the poles of the elements of $A_2(x)$ and congruent to them. Moreover these functions are represented asymptotically by $s_{ij:12}(x)$ throughout the entire plane, with respect to x to the left of any line parallel to the axis of imaginaries, and with respect to v to the right of such a line.

But the sequence $p^{(m)}_{ij:12}(x)$, $m = 1, 2, \cdots$ is identical with the sequence of determinants $u^{(m)}_{ij}(x)$ formed from the elements of the first two columns which are in the ith and jth rows of $P_m(x)$. In fact the first member of both sequences is $t_{ij:12}(x)$, and in general $p^{(m+1)}_{ij:12}(x)$ is derived from $p^{(m)}_{ij:12}(x)$ just as $u^{(m+1)}_{ij}(x)$ is derived from $u^{(m)}_{ij}(x)$, namely we have

$$p^{(m+1)}_{ij:12}(x) = \sum_{k,\,l} a_{ij:kl}(x-1)p^{(m)}_{kl:12}(x-1),$$

$$u^{(m+1)}_{ij}(x) = \begin{vmatrix} p^{(m+1)}_{i1}(x), & p^{(m+1)}_{i2}(x) \\ p^{(m+1)}_{j1}(x), & p^{(m+1)}_{j2}(x) \end{vmatrix} = \begin{vmatrix} \sum_{\tau=1}^{n} a_{i\tau}(x-1)p^{(m)}_{\tau 1}(x-1), & \sum_{\tau=1}^{n} a_{i\tau}(x-1)p^{(m)}_{\tau 2}(x-1) \\ \sum_{\tau=1}^{n} a_{j\tau}(x-1)p^{(m)}_{\tau 1}(x-1), & \sum_{\tau=1}^{n} a_{j\tau}(x-1)p^{(m)}_{\tau 2}(x-1) \end{vmatrix}$$

$$= \sum_{k,\,l} a_{ij:kl}(x-1)u^{(m)}_{kl}(x-1).$$

Thus the theorem is true for $\lambda = 2$ also. Strictly speaking, we have only obtained functions $u_{ij}(x)$ for $i < j$; but we have obviously such functions for all i and j; these may be obtained by means of the equation $u_{ij}(x) = -u_{ji}(x)$. In like manner the theorem can be demonstrated for the remaining values of λ.

The existence of the functions $v_{ij\ldots l}(x)$ and their properties may be established in the same way. The rôle of right and left in the x-plane will be interchanged and the equation (4)′ replaces (4).

The functions $u_{ij\ldots l}(x)$ and $v_{ij\ldots l}(x)$ will be called the *determinant limits* on account of their origin.

In the case that the restrictive hypothesis is satisfied and all the roots ρ of the characteristic equation are equal, Theorem I supplies a complete existence theorem, every column of elements of $P_m(x)$ and $Q_m(x)$ converging to an analytic solution of (1). These conditions are met if one has

$$a_{ij}(x) = \delta_{ij} + \frac{a_{ij}^{(2)}}{x^2} + \frac{a_{ij}^{(3)}}{x^3} + \cdots.$$

In general however Theorem I supplies only one solution directly, and that is furnished by

$$g_1(x) = u_1(x),\ g_2(x) = u_2(x),\ \cdots,\ g_n(x) = u_n(x).$$

This set of functions must be a solution since the definition of $u_i(x)$ gives us

$$u_i(x+1) = \lim_{m=\infty} \sum_{\tau,\ldots,\eta=1}^{n} a_{i\tau}(x)\cdots a_{\nu\eta}(x-m)\,t_{\eta 1}(x-m)$$

$$= \sum_{\tau=1}^{n} a_{i\tau}(x)\,u_\tau(x).$$

§ 2. The Solutions Associated with the Determinant Limits.

The functions $u_i(x)$, $u_{ij}(x)$, $\cdots$ which have been obtained behave as though they were constituent determinants of a matrix of solutions of (2).

Theorem II. *There exist n solutions $g_{1j}(x), g_{2j}(x), \cdots, g_{nj}(x)\,[h_{1j}(x), h_{2j}(x), \cdots, h_{nj}(x)]\,(j = 1, 2, \cdots, n)$ of (1) over any discrete set of values congruent to a given value x_0 of x, such that every λ-rowed determinant formed from the first [last] λ-columns of the matrix $G(x)[H(x)]$ and the ith, jth, $\cdots$, lth rows will equal $u_{ij\ldots l}(x)\,[v_{ij\ldots l}(x)]$ at these points. The first solution $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)\,[h_{1n}(x), h_{2n}(x), \cdots, h_{nn}(x)]$ is uniquely determined, the second solution $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)\,[h_{1,n-1}(x), h_{2,n-1}(x), \cdots, h_{n,n-1}(x)]$ is determined up to an additive term obtained by multiplying all the corresponding elements of the first solution by an arbitrary function of period 1, and so on, any one of these solutions being determined up to an additive term linear in the corresponding elements of the preceding solutions with arbitrary multipliers of period 1.*

Proof. The theorem is true in so far as it relates to $\lambda = 1$ since $u_1(x)$, $u_2(x), \cdots, u_n(x)$ has been seen to give a solution of (1). We may write then

$$g_{11}(x) = u_1(x),\ g_{21}(x) = u_2(x),\ \cdots,\ g_{n1}(x) = u_n(x).$$

Next we shall demonstrate that there exists some solution $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)$ such that for all i and j

$$(22) \qquad \begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x) \end{vmatrix} = u_{ij}(x).$$

For this purpose we shall recur to the sequence of elements of $P_m(x)$. By means of the identity

$$\begin{vmatrix} p_{i1}^{(m)}(x) & p_{i2}^{(m)}(x) & p_{i1}^{(m)}(x) \\ p_{j1}^{(m)}(x) & p_{j2}^{(m)}(x) & p_{j1}^{(m)}(x) \\ p_{k1}^{(m)}(x) & p_{k2}^{(m)}(x) & p_{k1}^{(m)}(x) \end{vmatrix} = 0,$$

we infer, by developing according to the elements of the third column and letting m become infinite, that

$$u_{ij}(x)u_k(x) - u_{ik}(x)u_j(x) + u_{jk}(x)u_i(x) = 0 \qquad (i, j, k = 1, 2, \cdots, n).$$

The n quantities $g_{12}(x_0), g_{22}(x_0), \cdots, g_{n2}(x_0)$ uniquely determine a solution of (1) on the set of points congruent to x_0. Choose these n quantities so that (22) is satisfied for some fixed i such that $g_{i1}(x_0) \neq 0$, and for every j. All the rest of the equations (22) will then be satisfied, as is evident from the preceding equation. This gives us

$$\begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x) \end{vmatrix} g_{k1}(x) - \begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{k1}(x) & g_{k2}(x) \end{vmatrix} g_{j1}(x) + u_{jk}(x)g_{i1}(x) = 0$$

for $x = x_0$, which reduces to

$$u_{jk}(x) = \begin{vmatrix} g_{j1}(x) & g_{j2}(x) \\ g_{k1}(x) & g_{k2}(x) \end{vmatrix}$$

for $x = x_0$. Hence $g_{ij}(x)$ and $u_{ij}(x)$ are equal at $x = x_0$ for all i and j. But they are both constituent elements of a solution of (21), and therefore will agree at all congruent points, since only one solution of that equation exists on the discrete set of points congruent to x_0 whose elements have given values at this point. It is necessary to make an exception of points congruent to the poles of $A(x)$ or $A^{-1}(x-1)$, and such points are excluded.

The method by which this second solution was obtained clearly allows us to add to each element $g_{i2}(x)$ a term of the form $p(x)g_{i1}(x)$, where $p(x)$ is of period 1.

One may now in an entirely analogous way define a solution $g_{13}(x), g_{23}(x), \cdots, g_{n3}(x)$ on the set of points congruent to x_0. The identities to be employed in this case are

$$u_{ijk}(x)u_l(x) - u_{ijl}(x)u_k(x) + u_{ikl}(x)u_j(x) - u_{jkl}(x)u_i(x) = 0.$$

The function $g_{i3}(x)$ is only determined up to an additive term

$$p(x)g_{i1}(x) + q(x)g_{i2}(x).$$

By proceeding thus the theorem may be proved in a succession of steps in so far as it relates to the elements $g_{ij}(x)$.

The part of the theorem that relates to the elements $h_{ij}(x)$ is proved in an entirely similar way.

The solutions $g_{1j}(x), \cdots, g_{nj}(x)$ $[h_{1j}(x), \cdots, h_{nj}(x)]$ may be said to be the solutions *associated* with the determinant limits $u_i(x)$, $u_{ij}(x)\cdots$ $[v_i(x), v_{ij}(x)\cdots]$.

It is important to obtain explicit formulas for these functions $g_{ij}(x)$, *and we shall show that it is possible to obtain the solutions* $g_{1j}(x), g_{2j}(x), \cdots, g_{nj}(x)$ *for* $j = 1, 2, \cdots, n$ *in terms of* $u_i(x), u_{ij}(x), \cdots, u_{12\cdots n}$, *and the operation* $\sum$ *defined by the condition*

$$\sum_{t=x+1} \phi(t) - \sum_{t=x} \phi(t) = \phi(x).$$

The functions $g_{11}(x), \cdots, g_{n1}(x)$ are equal respectively to $u_1(x), \cdots, u_n(x)$. It will accordingly be sufficient to obtain formulas for $g_{1k}(x), \cdots, g_{nk}(x)$ in terms of the solutions $g_{1j}(x), \cdots, g_{nj}(x)(j = 1, 2, \cdots, k-1)$ and the determinant limits.

Form the determinant

$$(23) \quad \theta^{(k)}(x) = \begin{vmatrix} g_{11}(x) & g_{12}(x) & \cdots & g_{1k}(x) \\ g_{11}(x+1) & g_{12}(x+1) & \cdots & g_{1k}(x+1) \\ \cdots & \cdots & \cdots & \cdots \\ g_{11}(x+k-1) & g_{12}(x+k-1) & \cdots & g_{1k}(x+k-1) \end{vmatrix}.$$

If now in place of the elements $g_{1j}(x+m)$ $(m > 0)$ we insert their value as obtained from (1)

$$(24) \quad g_{1j}(x+m) = \sum_{\tau, \sigma, \cdots, \theta=1}^{n} a_{1\tau}(x+m-1)a_{\tau\sigma}(x+m-2)\cdots a_{\eta\theta}(x)g_{\theta j}(x),$$

in terms of the functions $g_{1j}(x), \cdots, g_{nj}(x)$, we see that the above determinant may be expressed as a sum of terms each of which contains as factor a determinant which reduces to one of the determinant limits of order k. In fact we have obviously

$$(25) \quad \theta^{(k)}(x) = \sum_{\tau, \sigma, \cdots=1}^{n} \left\{ a_{1\tau}(x) \left[\sum_{\tau=1}^{n} a_{1\tau}(x+1)a_{\tau\sigma}(x) \right] \cdots \right\} u_{\tau\sigma\cdots}(x).$$

Hence this determinant is to be regarded as a known function.

But the identity (23) may also be looked upon as a non-homogeneous linear difference equation of order $k-1$ in $g_{1k}(x)$, and may be written

$$(26) \quad \theta^{(k-1)}(x)g_{1k}(x+k-1) + \psi(x)g_{1k}(x+k-2) + \cdots \pm \theta^{(k-1)}(x+1)g_{1k}(x) = \theta^{(k)}(x).$$

The associated reduced homogeneous equation obtained by equating the left

hand member to zero has the solutions [see (23)] $g_{11}(x), \cdots, g_{1,k-1}(x)$, which are linearly independent unless $\theta^{(k-1)}(x)$ is zero. By a method analogous to that of variation of parameters for ordinary linear differential equations, let us put

$$g_{1k}(x) = \sum_{j=1}^{k-1} g_{1j}(x) l_j(x) \tag{27}$$

and determine $l_1(x), l_2(x), \cdots, l_{k-1}(x)$ by the equations

$$\begin{aligned} \sum_{j=1}^{k-1} g_{1j}(x+1)\Delta l_j(x) &= 0, \\ \sum_{j=1}^{k-1} g_{1j}(x+2)\Delta l_j(x) &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \sum_{j=1}^{k-1} g_{1j}(x+k-1)\Delta l_j(x) &= \frac{\theta^{(k)}(x)}{\theta^{(k-1)}(x)}, \end{aligned} \tag{28}$$

where as usual $\Delta\phi$ denotes the difference $\phi(x+1)-\phi(x)$. There are here $k-1$ linear equations to determine the $k-1$ unknowns $\Delta l_1(x), \cdots, \Delta l_{k-1}(x)$, and their determinant is precisely $\theta^{(k-1)}(x+1)$. Unless this determinant is identically zero, we have accordingly

$$\Delta l_j(x) = \frac{\theta^{(k)}(x) m_{j,k-1}(x)}{\theta^{(k-1)}(x)\theta^{(k-1)}(x+1)},$$

in which $m_{j,k-1}(x)$ is the cofactor of the element in the jth column and last row of the determinant $\theta^{(k-1)}(x+1)$:

$$\pm \begin{vmatrix} g_{11}(x+1) & \cdots & g_{1,j-1}(x+1) & g_{1,j+1}(x+1) & \cdots & g_{1,k-1}(x+1) \\ g_{11}(x+2) & \cdots & g_{1,j-1}(x+2) & g_{1,j+1}(x+2) & \cdots & g_{1,k-1}(x+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{11}(x+k-1) & \cdots & g_{1,j-1}(x+k-1) & g_{1,j+1}(x+k-1) & \cdots & g_{1,k-1}(x+k-1) \end{vmatrix}. \tag{29}$$

The resulting expression for $g_{1k}(x)$ is

$$g_{1k}(x) = \sum_{j=1}^{k-1} g_{1j}(x) \sum_{t=x} \frac{\theta^{(k)}(t) m_{j,k-1}(t)}{\theta^{(k-1)}(t)\theta^{(k-1)}(t+1)}. \tag{30}$$

It remains to show that $g_{1k}(x)$ as thus defined is a solution of (26). We have

$$\begin{aligned} g_{1k}(x) &= \sum_{j=1}^{k-1} g_{1j}(x) l_j(x), \\ g_{1k}(x+1) &= \sum_{j=1}^{k-1} [(g_{1j}(x+1) l_j(x) + g_{1j}(x+1)\Delta l_j(x)] \\ &= \sum_{j=1}^{k-1} g_{1j}(x+1) l_j(x), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{aligned} \tag{31}$$

$$g_{1k}(x+k-1)=\sum_{j=1}^{k-1}\left[(g_{1j}(x+k-1)l_j(x)+g_{1j}(x+k-1)\Delta l_j(x)\right]$$
$$=\sum_{j=1}^{k-1}g_{ij}(x+k-1)l_j(x)+\frac{\theta^{(k)}(x)}{\theta^{(k-1)}(x)}.$$

Hence, if we multiply the last of these equations by $\theta^{(k-1)}(x)$, the one which precedes it by $\psi(x)$ and so on, and add corresponding members, the left hand member of the resulting equation will be precisely the left hand member of (26). Also all the terms except the last of the right hand member reduce to zero by reason of the fact that $g_{11}(x), \cdots, g_{1,k-1}(x)$ are solutions of the reduced equation, and there remains only a term $\theta^{(k)}(x)$. Therefore $g_{1k}(x)$ is the desired solution: furthermore the formula (30) does not determine $g_{1k}(x)$ uniquely but only up to a sum of terms

$$p_1(x)g_{11}(x)+\cdots+p_{k-1}(x)g_{1,k-1}(x),$$

where $p_1(x), \cdots, p_{k-1}(x)$ are arbitrary periodic multipliers since the operation $\sum_{t=x}$ is determined only up to an additive periodic term. But the result stated in Theorem II shows that $g_{1k}(x)$ has precisely this degree of indeterminateness, and hence *the most general determination of* $g_{1k}(x)$ *is furnished by* (30) *unless* $\theta^{(k-1)}(x)$ *is identically zero,* and this is an obvious case of failure.

When a choice of $g_{1k}(x)$ has been made, the remaining functions $g_{2k}(x), \cdots, g_{nk}(x)$ which make up the kth solution are determined. In fact the equations (24) for $m=1, 2, \cdots, n$ form n linear equations in $g_1(x), \cdots, g_n(x)$ of determinant not zero, since in the contrary case we should obtain a linear homogeneous relation between $g_1(x+1), \cdots, g_1(x+n)$, which contradicts the condition placed on $g_1(x)$ at the outset that it satisfies an equation of the nth order. By solving these equations we obtain

$$(32)\quad g_i(x)=\sigma_{i1}(x)g_1(x+1)+\sigma_{i2}(x)g_1(x+2)+\cdots+\sigma_{in}(x)g_1(x+n) \qquad (i=1, 2, \cdots, n),$$

where the functions $\sigma_{ij}(x)$ are rational in x. These equations determine $g_2(x), \cdots, g_n(x)$ in terms of $g_1(x)$.

That none of functions $\theta^{(1)}(x), \cdots, \theta^{(n)}(x)$ can be identically zero may be proved as follows: In the first place $\theta^{(k)}(x)$ has a definite asymptotic form in the entire x-plane, namely that obtained by replacing $u_{\tau\sigma\ldots}(x)$ by $s_{\tau\sigma\ldots}(x)$ in the right hand member of (25). This is a consequence of Theorem I. But this form is

$$\begin{vmatrix} s_{11}(x) & \cdots & s_{1k}(x) \\ s_{11}(x+1) & \cdots & s_{1k}(x+1) \\ \cdot & \cdots & \cdot \\ s_{11}(x+k-1) & \cdots & s_{1k}(x+k-1) \end{vmatrix},$$

as may be shown readily by means of (23). Recalling now the explicit form of the series $s_{ij}(x)$, we obtain

$$(33)\qquad \theta^{(k)}(x) \sim x^{k\mu x}(\rho_1 e^{-\mu})^x \cdots (\rho_k e^{-\mu})^x x^{r_1+\cdots+r_k}\left(d_k + \frac{d_k^{(1)}}{x} + \cdots\right).$$

Since $\theta^{(k)}(x)$ is represented asymptotically by this expression with respect to x to the left of any line parallel to the axis of imaginaries, and with respect to v to the right of such a line (Theorem I), it is obvious that unless $d_k, d_k^{(1)}, \cdots$ are all zero, $\theta^{(k)}(x)$ does not vanish at a great distance from the positive axis of reals. But these constants can not all vanish. Otherwise the above determinant vanishes (formally) and $s_{11}(x), \cdots, s_{1k}(x)$ are linearly dependent, and accordingly, by (32), $s_{i1}(x), \cdots, s_{ik}(x)$ $(i = 1, 2, \cdots, n)$, are linearly dependent, in contradiction to (7). Thus (30) is a valid formula.

§ 3. The Intermediate Associated Solutions.

The application of the formulas (30) *and* (32) *leads us immediately to a set of intermediate associated solutions*

$$g'_{1k}(x), \cdots, g'_{nk}(x) \qquad (k=1, 2, \cdots, n),$$

analytic above and below lines $A\infty$ and $B\infty$ parallel to the axis of reals, and which have the property that

$$g'_{1k}(x) \sim s_{1k}(x), \cdots, g'_{nk}(x) \sim s_{nk}(x) \qquad (k=1, 2, \cdots, n),$$

with respect to x in any left half plane and with respect to v in any right half plane.

We take

$$g'_{11}(x) = u_1(x), \cdots, g'_{n1}(x) = u_n(x),$$

and this first solution has the properties stated, by Theorem 1. We now define the solutions $g'_{12}(x), \cdots, g'_{n2}(x)$; $g'_{13}(x), \cdots, g'_{n3}(x)$; $\cdots$ in succession by choosing the summation operation Σ on $\phi(x)$ to be defined as

$$-\phi(x) - \phi(x+1) - \cdots$$

and employing formulas (30) and (32). It will suffice to show that a solution $g'_{1j}(x), \cdots, g'_{nj}(x)$ for $j = k$ will exist and have these properties under the hypothesis that the same is true for $j = 1, 2, \cdots, k-1$. The formula (30), which serves to define $g'_{1k}(x)$ if $|\rho_{k-1}| > |\rho_k|$, is then

$$(34)\qquad g'_{1k}(x) = -\sum_{j=1}^{k-1} g'_{1j}(x) \sum_{\nu=0}^{\infty} \frac{\theta^{(k)}(x+\nu) m'_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\theta^{(k-1)}(x+\nu+1)}.$$

We have written $m'_{j,k-1}(x)$ to stand for what $m_{j,k-1}(x)$ becomes when g'_{ij} replaces g_{ij}. We shall first prove that the $k-1$ series which appear here are

convergent. To this end we need to compute the asymptotic form of the typical term of the above series

$$\frac{\theta^{(k)}(x)\,m'_{j,\,k-1}(x)}{\theta^{(k-1)}(x)\theta^{(k-1)}(x+1)},$$

and this is done by merely substituting the known asymptotic forms for the θ's as given in (33) and at the same time the known form of the elements $m'_{j,\,k-1}$. This latter [see (29)] is clearly given by

$$\begin{aligned} m'_{j,\,k-1}(x) \sim\ & x^{(k-2)\mu x}(\rho_1 e^{-\mu})^x \cdots (\rho_{j-1}e^{-\mu})^x (\rho_{j-1}e^{-\mu})^x \\ & \cdots (\rho_{k-1}e^{-\mu})^x x^{r_1+\cdots+r_{j-1}+r_{j+1}+\cdots+r_{k-1}} \left\{ m_j + \frac{m_j^{(1)}}{x} + \cdots \right\} \end{aligned} \tag{35}$$

as a result of our hypothesis. Combining (35) and (33) we find for the above term the asymptotic formula

$$\frac{\theta^{(k)}(x)\,m'_{j,\,k-1}(x)}{\theta^{(k-1)}(x)\theta^{(k-1)}(x+1)} \sim \left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j} x^{s_j} \left\{ \eta_j + \frac{\eta_j^{(1)}}{x} + \cdots \right\}. \tag{36}$$

The integer s_j is zero unless $d_{k-1} = 0$.

Since we have $|\rho_j| > |\rho_k|$ for $j < k$ except when $|\rho_{k-1}| = |\rho_k|$ [see (8)], the terms in (34) clearly diminish in approximately geometrical ratio for v large enough, and in consequence the series converge uniformly in the vicinity of such points. Thus $g'_{1k}(x)$ has been actually defined and is analytic above and below suitable lines $A\infty$ and $B\infty$.

By the aid of relations (36) we may now determine the asymptotic form of each of the series in (34). In the first place, if each term of the jth series is replaced by its asymptotic formula, the series becomes

$$\sum_{\nu=0}^{\infty} \left(\frac{\rho_k}{\rho_j}\right)^{x+\nu} (x+\nu)^{r_k-r_j} (x+\nu)^{s_j} \left(\eta_j + \frac{\eta_j^{(1)}}{x+\nu} + \cdots \right).$$

The error committed in the final bracketed series by breaking off at the lth term is of order $(x+\nu)^{-l}$ if x lies to the left of the axis of imaginaries, and of order v^l if x lies to the right of this line, since the imaginary coefficient v of $x+\nu$ is the same as the imaginary coefficient of x. These bracketed series are, of course, finite above and below suitable lines $A\infty$ and $B\infty$.

Now separate the series into two parts: first the sum of the first m terms in which $\nu < \frac{1}{2}|x|$, secondly, the remaining terms.

In this first set of terms we have

$$(x+\nu)^\rho = x^\rho \left(1 + \frac{\rho\nu}{x} + \cdots \right),$$

where the error committed by breaking off the series at the kth term is of the order $(\nu/x)^k$. Hence the first m terms of the above series may be written

$$\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu\left[\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots+M_j(x)\left(\frac{\nu}{x}\right)^k\right],$$

where $M_j(x)$ is limited. Now the terms involving $M_j(x)$ are of the order x^{-k}, since

$$\left|\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu M_j(x)\left(\frac{\nu}{x}\right)^k\right| \leqq \left|\frac{M}{x^k}\sum_{\nu=0}^{\infty}\left(\frac{\rho_k}{\rho_j}\right)^\nu \nu^k\right|, \ M \text{ a fixed constant.}$$

As m becomes large we have also

$$\sum_{\nu=0}^{m}\left(\frac{\rho_k}{\rho_j}\right)^\nu \psi \sim \frac{\rho_j\psi}{\rho_j-\rho_k}$$

o all powers of x since the difference in absolute value is

$$\left|\frac{\left(\frac{\rho_k}{\rho_j}\right)^{m+1}\psi}{1-\frac{\rho_k}{\rho_j}}\right| < \left|\frac{\left(\frac{\rho_k}{\rho_j}\right)^{\left|\frac{x}{2}\right|}\psi}{1-\frac{\rho_k}{\rho_j}}\right|.$$

Hence the first m terms have the asymptotic form

$$\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\left\{\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots\right\} \tag{37}$$

with respect to x for x to the left of the axis of imaginaries, or similarly to the left of any parallel line. The terms after the mth diminish in approximately geometrical ratio for $|v|$ sufficiently large, and all contain a very small multiplier $(\rho_k/\rho_j)^m$. Hence the second group of terms does not affect the above asymptotic form.

Substituting the above expression in (34) and at the same time the known asymptotic formulas $s_{1j}(x)$ for $g'_{1j}(x)$, we find

$$g'_{1k}(x)\sim\sum_{j=1}^{k-1}x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left\{s_{1j}+\frac{s_{1j}^{(1)}}{x}+\cdots\right\}$$
$$\times\left(\frac{\rho_k}{\rho_j}\right)^x x^{r_k-r_j}x^{s_j}\left\{\psi_j+\frac{\psi_j^{(1)}}{x}+\cdots\right\},$$

or multiplying out,

$$g'_{1k}(x)\sim(\rho_k e^{-\mu})^x x^{r_k}x^s\left\{\xi_{1k}+\frac{\xi_{1k}^{(1)}}{x}+\cdots\right\} \tag{38}$$

with respect to x in any left half plane.

By means of (32) we now obtain the asymptotic formulas for $g'_{2k}(x), \cdots, g'_{nk}(x)$. Thus we obtain n formal expansions which must of course be the constituent elements of some formal solution. The possibility that $\xi_{1k}, \xi_{1k}^{(1)}, \cdots$ are all zero must be excluded since from that fact we should infer that all the coefficients in the asymptotic expansion of $g'_{ik}(x)\,(i=1,2,\cdots,n)$, and consequently

of $\theta^{(k)}(x)$, were zero. The formal solution thus obtained must coincide with $s_{1k}(x), \cdots, s_{nk}(x)$ at least if $\rho_1, \cdots, \rho_n$ are distinct. For it is apparent that the only possible difference lies in a constant factor. By definition of $\theta^{(k)}(x)$ in terms of $g_{11}(x) = g'_{11}(x), \cdots, g_{1k} = g'_{1k}(x)$, and in virtue of the fact that the asymptotic form of $\theta^{(k)}(x)$ has been shown to be that obtained by replacing these functions by $s_{11}(x), \cdots, s_{1k}(x)$, it follows that this factor must be 1. Hence we have in this case $\xi_{ik} = s_{ik}$, $\xi^{(1)}_{ik} = s^{(1)}_{ik}, \cdots$. Under our restriction, these equations are algebraic identities persisting in the exceptional cases when $\rho_1, \cdots, \rho_n$ are not distinct also.

Thus we have demonstrated the existence of a kth solution having the desired properties provided that $|\rho_{k-1}|$ and $|\rho_k|$ are not equal.

We shall now indicate briefly the modifications that are necessary in the case $|\rho_{k-1}| = |\rho_k|$. Suppose that ρ_k is one of a group of roots

$$\rho_\alpha, \rho_{\alpha+1}, \cdots, \rho_\beta$$

of equal absolute values. If $\rho_k = \rho_\alpha$, we have seen how to construct the kth solution; if $\rho_k = \rho_{\alpha+1}$, one may invert the order of ρ_α and $\rho_{\alpha+1}$ in (8) and thus construct a $(k+1)$th solution possessing the requisite asymptotic properties, and so continue to the βth solution. It is however necessary to know that the solutions thus constructed are *associated* with $u_i(x), u_{ij}(x), \cdots$ as stated. According to the method used in § 2 there certainly exist αth, $(\alpha+1)$th, $\cdots$, βth solutions over congruent sets of points which satisfy *all* the limit determinant relations, obtained by interchanging the order of $\rho_\alpha, \cdots, \rho_\beta$ in (8); for the algebraic condition that this hold will be satisfied at all the approximations since these are actual determinants, and hence will be satisfied in the limit also. On this account at least certain determinations of the sum formula employed will give a kth associated solution. Since however this kth associated solution is determined up to a linear combination with periodic coefficients of the solutions before the αth, and the formula which gives it indicates a solution determined up to a linear combination with periodic coefficients of solutions before the αth, the general solution given by the sum formula is a kth associated solution. Thus the solutions, obtained in this exceptional case as indicated, are associated with the determinant limits.

It would be of considerable interest to determine the nature of these intermediate solutions near to the axis of reals.

Considerations of symmetry show that there exists a set of solutions $h'_{1j}(x), \cdots, h'_{nj}(x)$ $(j = 1, 2, \cdots, n)$ having the properties stated.

§ 4. The First and Second Principal Matrix Solutions.

We shall proceed now to prove the existence of certain two remarkable matrix solutions $G(x)$ and $H(x)$ of (4), and this will be done by means of the

formulas of § 2 and the intermediate solutions of § 3. It is clear that in order to define any matrix $G(x)$ it is merely necessary to choose properly the operations $\sum_{t=x}$ which occur in (30). We shall use a contour integral to effect these summations, similar to that which Carmichael has used, writing

$$\sum_{t=x}\phi(t) = -\int_L e^{2\pi\lambda\sqrt{-1}(x-t)}\frac{\phi(t)\,dt}{e^{2\pi\sqrt{-1}(x-t)}-1},$$

where λ is an integer and L is a path $\infty\, A\, B\, \infty$ like that in fig. 2. The contour AB is a simple curve which passes between x and $x-1$. The summation function $\phi(t)$ is supposed to be analytic except at points at a limited distance from the positive axis of reals and within the region $\infty\, A\, B\, \infty$. Furthermore $\phi(t)$ is supposed to tend toward zero along lines $A\infty$, $B\infty$ sufficiently distant from the axis of reals, and in such manner as to make the above integral absolutely and uniformly convergent in the vicinity of any x not congruent to a point of L. In this case the function represented by the integral will be analytic at a sufficient distance from the positive axis of reals, and will be independent of the particular lines $A\infty$, $B\infty$ or particular curve AB chosen. If we form the difference $\sum_{t=x+1}\phi(t) - \sum_{t=x}\phi(t)$ it will be given by an integral over $ABB'A'A$, the integrand being unchanged. But within this closed curve the integrand is analytic save for a pole at $t=x$; and by the calculus of residues this difference reduces to $\phi(x)$. Hence this is a possible determination of $\sum_{t=x}\phi(t)$.

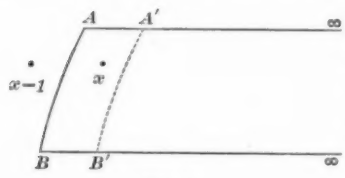

FIG. 2.

Having made this preliminary study of the operation $\sum_{t=x}$, we now proceed to the proof of

THEOREM III. *There exists a matrix solution $G(x)\,[H(x)]$ of (4) whose elements have the following properties:*

(*a*) *the functions $g_{ij}(x)\,[h_{ij}(x)]$ satisfy the conditions*

$$g_{i1}(x) = u_i(x),\quad \begin{vmatrix} g_{i1}(x) & g_{i2}(x) \\ g_{j1}(x) & g_{j2}(x)\end{vmatrix} = u_{ij}(x),\ \cdots$$

$$\left[\, h_{in}(x) = v_i(x),\quad \begin{vmatrix} h_{i,n-1}(x) & h_{in}(x) \\ h_{j,n-1}(x) & h_{jn}(x)\end{vmatrix} = v_{ij}(x),\ \cdots \right];$$

(*b*) *the functions $g_{ij}(x)[h_{ij}(x)]$ are analytic throughout the finite plane except possibly for poles to the right (left) of and congruent to the poles of the elements of $A(x)[A^{-1}(x-1)]$;*

(*c*) *in any left (right) half plane $g_{ij}(x)[h_{ij}(x)]$ is asymptotically represented by $s_{ij}(x)$ with respect to x.*

Proof. The proof of this theorem will be made by actually obtaining the solutions $g_{11}(x), \cdots, g_{n1}(x); \cdots; g_{1n}(x), \cdots, g_{nn}(x)$ in succession. The conditions of the theorem are evidently satisfied by taking the first solution to be

$$g_{11}(x) = u_1(x), \cdots, g_{n1}(x) = u_n(x).$$

It suffices therefore to exhibit a solution $g_{1k}(x), \cdots, g_{nk}(x)$ which satisfies the conditions of the theorem *under the hypothesis that the solutions* $g_{1j}(x)$, $\cdots, g_{nj}(x)$ $(j < k)$ *have been constructed satisfying all of these conditions.* This is the way in which the theorem will be demonstrated.

The formula (30) gives the most general determination of $g_{1k}(x)$ which is possible if $g_{1k}(x)$ is to be the first member of a kth solution satisfying restriction (a). We shall consider first the case $|\rho_{k-1}| > |\rho_k|$. We choose the operation $\sum_{t=x}$ as indicated above, and obtain

$$(39)\quad g_{1k}(x) = -\sum_{j=1}^{k-1} g_{1j}(x) \int_L e^{2\pi\lambda_{jk}\sqrt{-1}(x-t)} \frac{\theta^{(k)}(t)\, m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)} \frac{dt}{e^{2\pi\sqrt{-1}(x-t)} - 1};$$

the integer λ_{jk} *will be chosen to be the least integer as great as the real part of*

$$(40)\qquad \frac{1}{2\pi\sqrt{-1}} (\log \rho_k - \log \rho_j).$$

The functions $g_{2k}(x), \cdots, g_{nk}(x)$ are then obtained from (32).

The first step is to establish the fact that the summation functions in (39) satisfy the restriction on the summation function $\phi(t)$. Now $\theta^{(k)}(x)$, $\theta^{(k-1)}(x)$ are analytic save at points to the right of and congruent to the poles of elements $a_{ij}(x)$ by (25). Likewise, by the initial hypothesis and by definition of $m_{j,\,k-1}(x)$ as the cofactor of the element in the jth column and last row of $\theta^{(k-1)}(x+1)$ [see the form (29)], the singularities of $m_{j,\,k-1}(x)$ $(j = 1, 2, \cdots, k-1)$ are restricted in the same way. Hence the summation function

$$\phi(t) = \frac{\theta^{(k)}(t)\, m_{j,\,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}$$

will be analytic at all points sufficiently far from the positive axis of reals, provided that $\theta^{(k-1)}(t)$ does not vanish except at points within a certain distance of that axis. This fact is an immediate consequence of the relation (33) in which $d_k, d_k^{(1)}, \cdots$ have been seen not all to be zero. It remains to show that the function $\phi(t)$ tends to zero in such a way that the integrals appearing in (39) converge as stated for all positions of the lines $A\infty$ and $B\infty$ at a sufficient distance from the axis of the reals. In order to demonstrate this essential fact we make use of the intermediate solutions obtained in § 3. By Theorem II the general determination of the solutions $g_{1j}(x), \cdots, g_{nj}(x)$ associated with

the determinant limits is given by any one of them plus arbitrary periodic multiples of the preceding ones. Hence we have

$$g_{ij}(x) = g'_{i1}(x)\,p_{1j}(x) + \cdots + g'_{i,j-1}(x)\,p_{j-1,j}(x) + g'_{ij}(x) \qquad (41)$$
$$(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, k-1),$$

where the functions $p_{ij}(x)$ are periodic functions of period 1, analytic for x sufficiently distant from the axis of reals since the determinant $|g'_{ij}(x)| \sim |s_{ij}(x)|$ does not then vanish. Substitute these expressions (41), for $i = 1$, in $m_{j,k-1}(t)$. A reference to (29) makes it clear that the $m_{j,k-1}(t)$ may be written as a sum of products of periodic multipliers analytic for x sufficiently distant from the axis of reals and of factors $m'_{j,k-1}(t)$, where $m'_{j,k-1}$ signifies as before that which $m_{j,k-1}$ becomes when g' is put in place of g. Hence the integrals will converge in the manner desired if the expressions

$$\frac{\theta^{(k)}(t)\,m'_{j,k-1}(t)}{\theta^{(k-1)}(t)\,\theta^{(k-1)}(t+1)}$$

tend to zero in exponential fashion along $A\infty$ and $B\infty$. This is true if $A\infty$ and $B\infty$ are sufficiently distant from the axis of reals, in view of the asymptotic relationship (36) which holds with respect to v in the right half plane.

The function $g_{1k}(x)$ is therefore analytic at a sufficient distance from the positive axis of reals, and by (32) the same may be said of the remaining elements $g_{2k}(x), \cdots, g_{nk}(x)$ which constitute the kth solution. Working to the right across the plane by the aid of the original equations one proves readily that the only possible singularities of the functions of this solution will be poles congruent to the poles of the elements of $A(x)$ and to the right of them. Hence (b) is true for the kth solution. Also (a) holds in so far as it relates to this solution since every solution $g_{1k}(x), \cdots, g_{nk}(x)$ determined by (30) had this property.

Statement (c) has to do with the asymptotic nature of $g_{1k}(x), \cdots, g_{nk}(x)$. In order to prove (c) let us break up each integral in (39) into two others over contours L_1 and L_2 as follows: the contour L_1 is a fixed contour $\infty A_1 B_1 \infty$ (fig. 3). If x lies above $A_1\infty$, L_2 consists of a loop-circuit to ∞ which includes the points $x, x+1, \cdots$ within it but not $x-1, x-2, \cdots$. If x lies between $A_1\infty$ and $B_1\infty$, L_2 consists of a loop which includes $x, x+1, \cdots, x+l$ within it, where $x+l$ is the last of this series of points to the left of $A_1 B_1$. If x lies below $B_1\infty$, L_2 is a loop-circuit of the same nature as when x lies above $A_1\infty$. Any combination of paths L_1 and L_2 is clearly equivalent to the single path $\infty AB\infty$ of fig. 2. If x lies on $A\infty$ or $B\infty$ or on a line congruent to a point $A_1 B_1$ on the left, the two integrals are undefined.

Let us consider first the part of the integrals contributed by L_1. The integrands are periodic functions of x of period 1 and therefore the integrals will

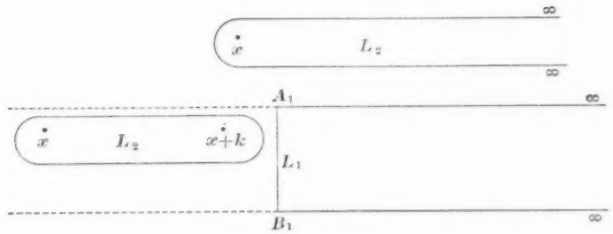

FIG. 3.

also be periodic since the contour L_1 is fixed. This part of the jth one of these integrals may be written as either of the products

$$e^{2\pi\lambda_{jk}\sqrt{-1}x}\int_{L_1} e^{-2\pi\lambda_{jk}\sqrt{-1}t}\,\frac{\theta^{(k)}(t)\,m_{j,k-1}(t)}{\theta^{(k-1)}(t)\theta^{(k-1)}(t+1)}\,\frac{dt}{e^{2\pi\sqrt{-1}(x-t)}-1},$$

$$e^{2\pi(\lambda_{jk}-1)\sqrt{-1}x}\int_{L_1} e^{-2\pi(\lambda_{jk}-1)\sqrt{-1}t}\,\frac{\theta^{(k)}(t)\,m_{j,k-1}(t)}{\theta^{(k-1)}(t)\theta^{(k-1)}(t+1)}\,\frac{dt}{1-e^{-2\pi\sqrt{-1}(x-t)}}.$$

It is apparent that in the first of these the second factor remains limited as v becomes positively infinite, inasmuch as $e^{2\pi\sqrt{-1}x}$ tends toward zero; likewise that in the second of these the second factor remains limited as v becomes negatively infinite. Hence the part of g_{1k} thus obtained is

$$(42)\qquad C_{L_1}=\begin{cases} g_{11}(x)e^{2\pi\lambda_{1k}\sqrt{-1}x}q_{1k}(x)+\cdots+g_{1,k-1}(x)e^{2\pi\lambda_{k-1,k}\sqrt{-1}x}q_{k-1,k}(x), \\ g_{11}(x)q_{1k}(x)+\cdots+g_{1,k-1}(x)q_{k-1,k}(x), \\ g_{11}(x)e^{2\pi(\lambda_{1k}-1)\sqrt{-1}x}q_{1k}(x)+\cdots+g_{1,k-1}(x)e^{2\pi(\lambda_{k-1,k}-1)\sqrt{-1}x}q_{k-1,k}(x), \end{cases}$$

according as x lies above $A_1\infty$, between $A_1\infty$ and $B_1\infty$, or below $B_1\infty$. The functions $q_{1k}(x), \cdots, q_{k-1,k}(x)$ are periodic and limited. This is obviously true unless x is in the vicinity of a point congruent to a point of the contour of integration. However it is true in this case also since each of the three expressions above is unchanged if the contour be varied continuously; of course the function $q_{ik}(x)$ is not one and the same analytic function in the three cases.

Next let us consider the contribution C_{L_2} yielded by the integrals taken over L_2. Let us suppose first that x lies above $A_1\infty$ or below $B_1\infty$. In this event the integrals in C_{L_2} may be evaluated as a sum of residues, each of which may be expressed in two ways, as indicated in the following formulas

$$(43)\qquad C_{L_2}=-\sum_{j=1}^{k-1}g_{1j}(x)\sum_{\nu=0}^{\infty}\left[\frac{\theta^{(k)}(x+\nu)\,m_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\theta^{(k-1)}(x+\nu+1)}\right]$$

or

$$C_{L_2}=-\sum_{j=1}^{k-1}g_{1j}(x)\sum_{\nu=0}^{\infty}\left[\frac{\theta^{(k)}(x+\nu)\,m'_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\theta^{(k-1)}(x+\nu+1)}\right].$$

The first form is the obvious one derived from equation (39). If we observe however that we have, by definition of $m_{j,k-1}(x)$,

$$\sum_{j=1}^{k-1} g_{1j}(x)m_{j,k-1}(x+\nu) = \begin{vmatrix} g_{11}(x+\nu+1)\cdots g_{1,k-1}(x+\nu+1) \\ g_{11}(x+\nu+2)\cdots g_{1,k-1}(x+\nu+2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g_{11}(x) \qquad\qquad \cdots g_{1,k-1}(x) \end{vmatrix},$$

and then write in place of g_{1j} their equals as given in (41), the resulting determinant may be simplified because the periodic coefficients are the same at $x+\nu+1, x+\nu+2, \cdots$ as at x. By a combination of successive columns the determinant reduces to a new determinant identical with the above except that g' replaces g, i. e., reduces to

$$\sum_{j=1}^{k-1} g'_{1j}(x)m'_{j,k-1}(x+\nu).$$

Thus the second of the above forms is obtained except that the two signs of summation will appear in inverted order. It is permissible to change this order if all the infinite series converge. This is certainly the case as the second form is precisely that which defined $g'_{1k}(x)$ in § 3 [see (34)]. Thus we have for x above $A_1\,\infty$ or below $B_1\,\infty$

$$C_{L_2} = g'_{1k}(x). \tag{44}$$

When x lies between $A_1\,\infty$ and $B_1\,\infty$ to the left of A_1B_1 we use a form like (43). We will agree to denote C_{L_2} by $g'_{1k}(x)$ for the sake of convenience, because then (44) holds in all cases and $g'_{1k}(x)$ *as thus defined will have the asymptotic form* s_{1k} *with respect to* x *in any left half plane, and with respect to* v *in any right half plane.* In § 3 these facts have already been proved for $g'_{1k}(x)$ as there defined, that is for x not between $A_1\,\infty$ and $B_1\,\infty$. We need therefore to prove that $g'_{1k}(x) \sim s_{1k}(x)$ with respect to x between $A_1\,\infty$ and $B_1\,\infty$ only. The discussion is entirely similar to a discussion that has been given in § 3. In the first place the integrals in the expression $g'_{1k}(x)$ for C_{L_2} may be evaluated as a sum of residues

$$g'_{1k}(x) = -\sum_{j=1}^{k-1} g_{1j}(x) \sum_{\nu=0}^{l} \frac{\theta^{(k)}(x+\nu)m_{j,k-1}(x+\nu)}{\theta^{(k-1)}(x+\nu)\theta^{(k-1)}(x+\nu+1)}, \tag{45}$$

where $x+l$ is the last point of the set $x, x+1, \cdots$ to the left of A_1B_1. If this equation be compared with (34) one sees that the right hand members differ in that a finite sum replaces an infinite series. Proceeding precisely as we did with (34) we arrive at the asymptotic relation $g'_{1k}(x) \sim s_{1k}(x)$ with respect to x by means of the hypothesis made that $g_{1j}(x), \cdots, g_{nj}(x)$ are asymptotically represented by $s_{1j}(x), \cdots, s_{nj}(x)$ with respect to x in any left half plane and

for $j = 1, 2, \cdots, k-1$. The sole modification consists in this that instead of having an *infinite* set of terms in which those after the mth (m the greatest integer less than $\frac{1}{2}|x|$) do not affect the asymptotic form, one has only a *finite* set of terms in which however those after the mth do not affect the asymptotic form.

The proof of (c) of the theorem in so far as it relates to $g_{1k}(x), \cdots, g_{nk}(x)$ may now be given. In the first place, for x above $A_1\infty$, (42) and (44) give

$$g_{1k}(x) \sim s_{11}(x)e^{2\pi\lambda_{1k}\sqrt{-1}x}q_{1k}(x) + \cdots + s_{1,k-1}(x)e^{2\pi\lambda_{k-1,k}\sqrt{-1}x}q_{k-1,k} + s_{1k}(x).$$

The dominating term will depend upon the dominating term of

$$e^{(2\pi\lambda_{1k}\sqrt{-1}+\log\rho_1)x}, \quad \cdots, \quad e^{(2\pi\lambda_{k-1,k}\sqrt{-1}+\log\rho_{k-1})x}, \quad e^{(\log\rho_k)x}$$

which begin the formal expansions. On dividing by the last of these, the exponents become

$$2\pi\sqrt{-1}\left(\lambda_{1k} + \frac{\log\rho_1 - \log\rho_k}{2\pi\sqrt{-1}}\right)x, \cdots, 2\pi\sqrt{-1}\left(\lambda_{k-1,k} + \frac{\log\rho_{k-1} - \log\rho_k}{2\pi\sqrt{-1}}\right)x, 0.$$

The bracketed expressions here all have a positive real part less than 1: this follows from the definition of $\lambda_{1k}, \cdots, \lambda_{k-1,k}$ and the fact that $|\rho_{k-1}| > |\rho_k|$. Hence the coefficients of x are represented by points in the upper half plane below the line $v = 2\pi$. Therefore if x lies in the second quadrant, all of these exponents have a negative real part, and the last term dominates, that is

$$g_{1k}(x) \sim s_{1k}(x)$$

for x above $A_1\infty$ and to the left of the axis of imaginaries, or likewise of any parallel line.

For x between $A_1\infty$ and $B_1\infty$ we have similarly

$$g_{1k}(x) \sim s_{11}(x)q_{1k}(x) + \cdots + s_{1,k-1}(x)q_{k-1,k}(x) + s_{1k}(x)$$

and the last term clearly dominates again by (8).

For x below $B_1\infty$, $g_{1k}(x)$ is represented by

$$s_{11}(x)e^{2\pi(\lambda_{1k}-1)\sqrt{-1}x}q_{1k}(x) + \cdots + s_{1,k-1}(x)e^{2\pi(\lambda_{k-1,k}-1)\sqrt{-1}x}q_{k-1,k}(x) + s_{1k}(x).$$

The dominating term depends on the exponents

$$2\pi\sqrt{-1}\left(\lambda_{1k}-1+\frac{\log\rho_1-\log\rho_k}{2\pi\sqrt{-1}}\right)x, \cdots, 2\pi\sqrt{-1}\left(\lambda_{k-1,k}-1+\frac{\log\rho_{k-1}-\log\rho_k}{2\pi\sqrt{-1}}\right)x, 0.$$

The coefficients of x will in this case clearly lie below the real axis and above the line $v = 2\pi$, and if x lies in the third quadrant, the last term $s_{1k}(x)$ again dominates.

Hence in any left half plane $g_{1k}(x)$ is asymptotically represented by $s_{1k}(x)$

with respect to x, and of course $g_{2k}(x), \cdots, g_{nk}(x)$ are at the same time represented by $s_{2k}(x), \cdots, s_{nk}(x)$ respectively, as follows from (32). Thus a solution $g_{1k}(x), \cdots, g_{nk}(x)$ exists possessing properties $(a), (b), (c)$ unless $|\rho_{k-1}| = |\rho_k|$, that is unless ρ_k is one of a group $\rho_a, \rho_{a+1}, \cdots, \rho_p$ of roots ρ of equal absolute value. The same argument which enabled us to treat this exceptional case in § 3 may now be applied here; that is, we may take ρ_k as the first of this group without violating (8) and construct a kth solution which satisfies $(a), (b), (c)$ corresponding to each of these values of ρ.

The second set of solutions $h_{1j}(x), \cdots, h_{nj}(x), (j = 1, 2, \cdots, n)$ may be obtained by considerations of symmetry.

There is one and only one solution $g_{1k}(x), \cdots, g_{nk}(x)$ $[h_{1k}(x), \cdots, h_{nk}(x)]$ $(k = 1, 2, \cdots, n)$ analytic except for poles and asymptotically represented by $s_{1k}(x), \cdots, s_{nk}(x)$ even to terms of the first order in any left (right) half plane. The matrix $G(x)\,[H(x)]$ with elements $g_{ij}(x)[h_{ij}(x)]$ is termed the *first* (*second*) *principal matrix solution* of (4).

In order to justify this definition it is necessary to show that there exists but one such solution $g_{1k}(x), \cdots, g_{nk}(x)[h_{1k}(x), \cdots, h_{nk}(x)]$. Suppose if possible that there exists a second solution $\bar{g}_{1k}(x), \cdots, \bar{g}_{nk}(x)$ with the properties stated. Now we can express $\bar{g}_{ik}(x)$ as follows:

$$\bar{g}_{ik}(x) = g_{i1}(x)p_1(x) + \cdots + g_{in}(x)p_n(x) \quad (i = 1, 2, \cdots, n),$$

where $p_1(x), \cdots, p_n(x)$ are periodic of period 1, inasmuch as $g_{1j}(x), \cdots, g_{nj}(x)$ $(j = 1, 2, \cdots, n)$ form a fundamental system of solutions. If we solve these equations for $p_i(x)$, we shall have

$$p_i(x) = \frac{\begin{vmatrix} g_{11}(x) & \cdots & \bar{g}_{1k}(x) & \cdots & g_{1n}(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{n1}(x) & \cdots & \bar{g}_{nk}(x) & \cdots & g_{nn}(x) \end{vmatrix}}{\begin{vmatrix} g_{11}(x) & \cdots & g_{1i}(x) & \cdots & g_{1n}(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{n1}(x) & \cdots & g_{ni}(x) & \cdots & g_{nn}(x) \end{vmatrix}}.$$

Consider now some definite period-strip formed by two lines parallel to the axis of imaginaries and one unit apart, and let x become infinite in this strip. On account of the known asymptotic form of the elements g and $\bar{g}$ in the strip to terms of the first order we find at once, if $i \neq k$,

$$p_i(x) \sim \left(\frac{\rho_k}{\rho_i}\right)^x x^{r_k - r_i}[0]$$

for x at either end of the strip. If we write $z = e^{2\pi\sqrt{-1}\,x}$, this may be written

$$p_i(x) \sim z^{\frac{1}{2\pi\sqrt{-1}}(\log \rho_k - \log \rho_i)}(\log z)^{r_k - r_i}[0].$$

The upper end of the strip corresponds to $z = 0$, and the lower end to $z = \infty$. The above relation shows that $p_i(x)$ has a zero of at least the order λ_{ik} at $z = 0$. Likewise, this relation shows that $p_i(x)$ has a pole of order $\lambda_{ik} - 1$ at most at $z = \infty$. Consequently $p(x)z^{-\lambda_{ik}}$ is analytic at $z = 0$ and vanishes at $z = \infty$. Moreover, by taking x far enough to the left, the original determinant expression for $p_i(x)$ is analytic since every element of the two determinants is analytic, and the denominator is asymptotic to the formal determinant $|s_{ij}(x)|$ which is not zero if x lies far to the left. Hence $p_i(x)z^{-\lambda_{ik}}$, for $i \neq k$, is analytic for all finite values of z and vanishes at $z = \infty$; it must be identically zero. We conclude that

$$g_{ik}(x) = g_{ik}(x)p_k(x).$$

Now let i be chosen so that s_{ik}, the constant of the principal term of $s_{ik}(x)$, is not zero, as is possible by (7). Consider any point x_0 and the points $x_0 - 1$, $x_0 - 2, \cdots$ to its left. As x takes on these values, $\bar{g}_{1k}(x)/g_{1k}(x)$ tends toward 1, and thus we see that $p_k(x)$ is 1 at the point x_0. That is, $\bar{g}_{1k}(x)$ and $g_{1k}(x)$ are one and the same function, in contradiction to our hypothesis.

If the functions $a_{ij}(x)$ had not been restricted to be rational, the only modification in §§ 1–4 would be that in the statement and proof of (*b*), Theorem III, other singularities than poles would enter.

§ 5. The Fundamental Periodic Functions.

Either of the two principal matrix solutions $G(x)$ and $H(x)$ of (4) may be expressed in terms of the other, e. g.,

$$G(x) = H(x)P(x), \tag{46}$$

in which the elements $p_{ij}(x)$ of $P(x)$ are periodic functions of period 1 which may be termed the *fundamental periodic functions*. We shall now determine the nature of these functions. The fact that will enable us to do this is that $G(x)$ and $H(x)$ are composed of elements of known asymptotic form in any period strip which may be taken to be one and the same in both cases.

On account of the gain in simplicity we shall make a preliminary transformation. Let

$$d(x) = (x - \alpha)(x - \beta)\cdots(x - \lambda)$$

be the least common denominator of the set of rational functions $a_{ij}(x)$, and write

$$\bar{g}_i(x) = \Gamma(x - \alpha)\Gamma(x - \beta)\cdots\Gamma(x - \lambda)g_i(x) \qquad (i = 1, 2, \cdots, n),$$

where $\Gamma(x)$ is the gamma function. The transformed system will be altered in that the common denominator has been removed. The formal solutions of the new system will be those of the old multiplied by a certain factor, and hence will satisfy the restrictions imposed in § 1.

The new coefficients are polynomials in x. It is then permissible to take all the functions $a_{ij}(x)$ as polynomials, say of degree μ or less, and this we shall do. The first principal matrix solution $G(x)$ is constituted of elements analytic throughout the finite plane, in this case, inasmuch as the elements $a_{ij}(x)$ have no poles in the finite plane.

From the equation (46) we have

$$P(x) = H^{-1}(x)\, G(x); \tag{47}$$

but if we take our period strip far enough to the right, the elements of $H(x)$ are analytic functions of x and the determinant of $H(x)$ will not vanish in the strip inasmuch as its asymptotic form is given by the determinant $|s_{ij}(x)|$. Consequently the elements $p_{ij}(x)$ are functions of $z = e^{2\pi\sqrt{-1}x}$ which are analytic except perhaps at $z = 0$ and $z = \infty$, corresponding to the upper and lower end of the period strip. In the case when the functions $a_{ij}(x)$ in (1) are merely rational it is clear that the periodic functions $p_{ij}(x)$ are analytic or have poles except at $z = 0$ or $z = \infty$.

It is necessary now to remove a certain ambiguity. The expression $s_{ij}(x)$ represents the elements $g_{ij}(x)$ and $h_{ij}(x)$ asymptotically in any right and left hand side of the plane respectively, and is a many-valued expression which is changed to

$$e^{2\pi\mu\sqrt{-1}x}\, e^{2\pi r_j\sqrt{-1}}\, s_{ij}(x) \tag{48}$$

after a complete positive circuit of $x = 0$ [see (6)]. Hence if the same determination of $s_{ij}(x)$ is chosen for $g_{ij}(x)$ and $h_{ij}(x)$ along the direction of the positive axis of imaginaries, the elements $g_{ij}(x)$ and $h_{ij}(x)$ will have a different asymptotic form along the direction of the negative axis of imaginaries, since in one case we pass from the positive axis of imaginaries to the negative by a positive partial circuit, and in the other by a negative partial circuit. (It must be remembered that $g_{ij}(x)$ and $h_{ij}(x)$ preserve their asymptotic form in any *left* and *right* half plane respectively.) Let the determination for $h_{ij}(x)$ be termed $s_{ij}(x)$. Then that of $g_{ij}(x)$ is obviously obtained by a complete positive circuit and is given by (48) in the direction of negative imaginaries.

In the upper half of the period strip $g_{ij}(x)$ and $h_{ij}(x)$ have a common asymptotic representation $s_{ij}(x)$, and this proves that

$$P(x) \sim S^{-1}(x)\, S(x).$$

But the elements of $s'_{ij}(x)$ and $s_{ij}(x)$ of $S^{-1}(x)$ and $S(x)$ are of the respective asymptotic forms

$$x^{\mu x}\rho_i^{-x}x^{-r_i}\left\{\sigma_{ij} + \frac{\sigma_{ij}^{(1)}}{x} + \cdots\right\}, \qquad x^{\mu x}\rho_j^{x}x^{r_j}\left\{s_{ij} + \frac{s_{ij}^{(1)}}{x} + \cdots\right\}$$

[see (9) and (6)], so that in consequence the relation holds

$$p_{ij}(x) \sim \rho_i^{-x} \rho_j^{x} x^{-r_i+r_j} \delta_{ij}.$$

Therefore, if we think of $p_{ij}(x)$ as a function of z, we have

$$\lim_{z=0} p_{ij}(x) z^{-\frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}} \left(\frac{\log z}{2\pi\sqrt{-1}} \right)^{r_i - r_j} = \delta_{ij}.$$

The single-valued analytic function $p_{ij}(x)$ of z is accordingly infinite only to a finite order at $z = 0$ and either has a pole there, or is analytic. For $i = j$ it is clear that $p_{ij}(x)$ is an entire function of z taking on the value 1 at $z = 0$, that is

$$p_{ii}(x) = 1 + c_{ii}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots. \tag{49}$$

For $i \neq j$ we have obviously

$$\lim_{z=0} p_{ij}(x) z^{-\lambda_{ij}} = \text{constant}$$

since λ_{ij} was defined as the least integer as great as the real part of

$$\frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}.$$

Hence $p_{ij}(x)$ has a zero of order λ_{ij}, at least, at $z = 0$, and may be written

$$p_{ij}(x) = e^{2\pi\lambda_{ij}\sqrt{-1}x}(c_{ij}^{(0)} + c_{ij}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots), \tag{50}$$

the coefficient of the exponential factor being an entire function of z.

It remains to consider the functions $p_{ij}(x)$ in the lower half of the period strip. Here the functions $g_{ij}(x)$ and $h_{ij}(x)$ have the respective asymptotic forms

$$e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} s_{ij}(x) \quad \text{and} \quad s_{ij}(x),$$

and the matrices $G(x)$ and $H(x)$ are asymptotically represented by

$$S(x) \quad \text{and} \quad S_1(x)$$

respectively where

$$S_1(x) = S(x)(e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij}).*$$

Hence we obtain

$$P(x) \sim S^{-1}(x) S_1(x) \quad \text{or} \quad [S^{-1}(x) S(x)](e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij})$$
$$\text{or} \quad \rho_i^{-x} \rho_j^{x} x^{-r_i+r_j}(e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}} \delta_{ij}).$$

Thus each function $p_{ij}(x)$ satisfies the condition

$$\lim_{z=\infty} p_{ij}(x) z^{-\mu - \frac{\log \rho_j - \log \rho_i}{2\pi\sqrt{-1}}} \left(\frac{\log z}{2\pi\sqrt{-1}} \right)^{r_i - r_j} e^{-2\pi r_j\sqrt{-1}} = \delta_{ij}.$$

* The notation (θ_{ij}) signifies a matrix with term θ_{ij} in the ith row and jth column.

We see again that $p_{ij}(x)$ becomes infinite only to a finite order at $z=\infty$. This demonstrates that $p_{ii}(x)$ is a polynomial in z and that $p_{ij}(x)$ $(i \neq j)$ is equal to $z^{\lambda_{ij}}$ multiplied by a polynomial in z.

In the case $i=j$ we have

$$\lim_{z=\infty} p_{ii}(x) z^{-\mu} = e^{2\pi r_i \sqrt{-1}},$$

so that the polynomial $p_{ii}(x)$ is of degree μ in z with leading coefficient $e^{2\pi r_i \sqrt{-1}}$. In the case $i \neq j$ we have

$$\lim_{z=\infty} p_{ij}(x) z^{-\mu-\lambda_{ij}+1} = \text{constant},$$

so that the polynomial coefficient of $z^{\lambda_{ij}}$ is of degree $\mu - 1$.

Thus we have proved the following theorem:

THEOREM IV. *Let $G(x)$ and $H(x)$ be the first and second principal matrix solutions of* (4) *in which the elements of $H(x)$ are polynomials of degree μ at most in x, and write*

$$G(x) = H(x) P(x).$$

The fundamental periodic functions $p_{ij}(x)$ which constitute $P(x)$ are in this case of the form

$$(51) \qquad \begin{aligned} p_{ii}(x) &= 1 + c_{ii}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots + e^{2\pi r_i \sqrt{-1}} e^{2\pi\mu\sqrt{-1}x}, \\ p_{ij}(x) &= e^{2\pi\lambda_{ij}\sqrt{-1}x}\left[c_{ij}^{(0)} + c_{ij}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots + c_{ij}^{(\mu-1)} e^{2\pi(\mu-1)\sqrt{-1}x}\right] \quad (i \neq j), \end{aligned}$$

in which λ_{ij} denotes the least integer as great as the real part of

$$\frac{1}{2\pi\sqrt{-1}}(\log \rho_j - \log \rho_i).$$

The simplest illustration is given by the difference equation for the Gamma function, when one has

$$g(x) = e^{\pi\sqrt{-1}x} \frac{-2\pi\sqrt{-1}}{\Gamma(1-x)}, \quad h(x) = \Gamma(x), \quad p(x) = 1 - e^{2\pi\sqrt{-1}x}.$$

A similar analysis of the nature of the functions $p_{ij}(x)$ at $z=0$ and $z=\infty$ may be made in the more general case when the functions $a_{ij}(x)$ are not restricted to be rational functions, and shows that $p_{ij}(x)$ is analytic or has a pole at $z=0$ and $z=\infty$ with leading terms of the same form as those of (51) at these points. If the $a_{ij}(x)$ are rational, it is obvious that *the functions $p_{ij}(x)$ are rational in $e^{2\pi\sqrt{-1}x}$*; we shall not stop to investigate further the structure of these functions.

§ 6. The Principal Matrix Solutions in the Entire Plane.

The results of the preceding paragraph make it possible to discuss the asymptotic form of the elements of $G(x)$ [and likewise of $H(x)$] in the entire complex plane. In the first place it has been shown that

$$g_{ij}(x) \sim s_{ij}(x)$$

along any ray from $x = 0$ in the second or third quadrants. To investigate the asymptotic form along rays in the first and fourth quadrants we make use of the equation

$$(52) \quad G(x) = H(x)P(x) \quad \text{or} \quad g_{ij}(x) = \sum_{\tau=1}^{n} h_{i\tau}(x)p_{\tau j}(x) \quad (i, j = 1, 2, \cdots, n),$$

in which the elements $h_{ij}(x)$ are of known asymptotic form $s_{ij}(x)$ in any right half plane, and the elements $p_{ij}(x)$ are known functions of x.

Consider now a ray from $x = 0$ in the first quadrant and lying above the axis of reals. Along this ray, by equations (51), we shall have

$$p_{ij}(x) \sim e^{2\pi\lambda_{ij}\sqrt{-1}x} c_{ij}^{(0)}, \quad \text{where } \lambda_{ii} = 0, \quad c_{ii}^{(0)} = 1.$$

Thus the preceding equation for $g_{ij}(x)$ leads to

$$(53) \qquad g_{ij}(x) \sim s_{i1}(x)e^{2\pi\lambda_{1j}\sqrt{-1}} c_{1j}^{(0)} + \cdots + s_{in}(x)e^{2\pi\lambda_{nj}\sqrt{-1}x} c_{nj}^{(0)},$$

a formula which determines the asymptotic form in the first quadrant.

To determine the dominant term we need to compare the relative magnitude of the terms

$$\rho_1^x e^{2\pi\lambda_{1j}\sqrt{-1}x}, \cdots, \rho_j^x, \cdots, \rho_n^x e^{2\pi\lambda_{nj}\sqrt{-1}x}.$$

If these be divided by ρ_j^x, the logarithms become

$$2\pi\sqrt{-1}\left(\frac{\log\rho_1 - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{1j}\right)x, \cdots, 0x, \cdots,$$
$$2\pi\sqrt{-1}\left(\frac{\log\rho_n - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{nj}\right)x.$$

By the very definition of λ_{ij}, the coefficients $\epsilon_{1j}, \cdots, \epsilon_{jj}=0, \cdots, \epsilon_{nj}$ of x in these expressions will lie in the strip $0 \leqq v < 2\pi$. Let $P_1, \cdots, P_n$ be the points which represent these coefficients in the complex plane. On account of (8) these coefficients will have decreasing real parts and the points $P_1, \cdots, P_n$ are ordered from right to left. As $\arg x$ increases from 0 to $\frac{1}{2}\pi$ the above logarithms maintain their relative positions but are rotated about $x = 0$ through the same angle. The dominant term along the ray $\arg x = \tau$ is that one for which the corresponding point is farthest to the right (i. e., the logarithm has the greatest real part) when a rotation τ has been effected. None of the terms after the j-th

can dominate since the corresponding points are to the left of $P_j = 0$ for all values of $\tau < \pi/2$.

Draw the convex broken line $P_1 P_\sigma \cdots P_j$ below which none of the points $P_1, \cdots, P_j$ lie and whose vertices are at some of these points. If the plane be rotated through an angle τ, it is clear that these vertex-points, and only these, will correspond to dominant terms. Let the acute angles which the successive sides of $P_1 P_\sigma \cdots P_j$ make with the axis of reals be $\phi_1, \phi_\sigma, \cdots$. The *critical rays* along which the dominant term changes are then in order of increasing magnitude

$$\frac{\pi}{2} - \phi_1, \quad \frac{\pi}{2} - \phi_\sigma, \quad \cdots.$$

For small values of arg x the dominant term is clearly given by

$$s_{i1}(x)\, e^{2\pi\lambda_{1j}\sqrt{-1}\,x} c_{1j}^{(0)},$$

and as the first critical ray is passed this changes to

$$s_{i\sigma}(x)\, e^{2\pi\lambda_{\sigma j}\sqrt{-1}\,x} c_{\sigma j}^{(0)}.$$

Along the ray the form is given by a combination of two or more terms, a fact which we shall express by saying that the change is *uniform* along the ray. A similar change takes place along each of the successive critical rays until the last is passed when the asymptotic form becomes $s_{ij}(x)$.

Likewise in the fourth quadrant the asymptotic nature is deduced from a relation

$$g_{ij}(x) \sim s_{i1}(x)\, e^{2\pi(\mu+\lambda_{1j}-1)\sqrt{-1}\,x} c_{1j}^{(\mu-1)} + \cdots + s_{ij}(x)\, e^{2\pi\mu\sqrt{-1}\,x} e^{2\pi r_j\sqrt{-1}} + \cdots + s_{in}(x)\, e^{2\pi(\mu+\lambda_{nj}-1)\sqrt{-1}\,x} c_{nj}^{(\mu-1)}.$$

The expressions $s_{i1}(x), \cdots, s_{in}(x)$ which appear here are the determinations for the functions $h_{i1}(x), \cdots, h_{in}(x)$ in the right half plane. Consequently the determination of $s_{ij}(x)$ obtained by a positive rotation from the first to the fourth quadrant along the negative axis of reals is this determination multiplied by the factor $e^{2\pi\mu\sqrt{-1}x} e^{2\pi r_j\sqrt{-1}}$. Hence in terms of the second determination

$$g_{ij}(x) \sim s_{i1}(x)\, e^{2\pi(\lambda_{1j}-1)\sqrt{-1}\,x} e^{-2\pi r_j\sqrt{-1}} c_{1j}^{(\mu-1)} + \cdots + s_{ij}(x) + \cdots + s_{in}(x)\, e^{2\pi(\lambda_{nj}-1)\sqrt{-1}x} e^{-2\pi r_j\sqrt{-1}} c_{nj}^{(\mu-1)}.$$

After division the corresponding logarithms are

$$2\pi\sqrt{-1}\left[\frac{\log\rho_1 - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{1j} - 1\right]x, \cdots, 0x, \cdots, 2\pi\sqrt{-1}\left[\frac{\log\rho_n - \log\rho_j}{2\pi\sqrt{-1}} + \lambda_{nj} - 1\right]x.$$

The coefficients of x are represented by points $Q_1, \cdots, Q_j = 0, \cdots, Q_n, 2\pi$ units below the corresponding points $P_1, \cdots, P_n$ (except for $P_j = Q_j = 0$). These points all lie in the strip $-2\pi \leqq v < 0$ and are given in order from right to left. As arg x decreases from 0 to $-\pi/2$, the figure formed by these points rotates through an angle $-\pi/2$, and the dominant terms correspond to the point farthest to the right. None of the terms after the jth can dominate.

In this case we are led to the following rule: draw a convex broken line $Q_1 Q_\theta \cdots Q_j$ above which none of the points $Q_1, Q_2, \cdots, Q_j$ lie and whose vertices are at these points. Let the acute angle which the successive sides $Q_1 Q_\theta \cdots Q_j$ make with the axis of x be $\psi_1, \psi_\theta, \cdots$. The critical rays along which the dominant term changes are then

$$-\frac{\pi}{2} + \psi_1, \quad -\frac{\pi}{2} + \psi_\theta, \quad \cdots$$

in order of decreasing magnitude. For small arg x the dominant term has the asymptotic form

$$s_{i1}(x) e^{2\pi(\lambda_{1j}-1)\sqrt{-1}x} e^{-2\pi r_j \sqrt{-1}} c_{1j}^{(\mu-1)},$$

and as the first critical ray is passed, this changes uniformly to

$$s_{i\theta}(x) e^{2\pi(\lambda_{\theta j}-1)\sqrt{-1}x} e^{-2\pi r_\theta \sqrt{-1}} c_{\theta_j}^{(\mu-1)},$$

and similar changes take place along the remaining critical rays until the last ray is passed, when the form is $s_{ij}(x)$.

THEOREM V. *Let the coefficients $a_{ij}(x)$ be polynomials of degree not greater than μ in x. Write*

$$2\pi\sqrt{-1}\left(\frac{\log \rho_i - \log \rho_j}{2\pi\sqrt{-1}} + \lambda_{ij}\right) = \epsilon_{ij}$$

where λ_{ij} is the least integer as great as the real part of the negative of the first term in parenthesis, and where $\rho_1, \rho_2, \cdots, \rho_n$ are the roots of the characteristic equation ordered according to decreasing absolute magnitude. Mark the two sets of points

$$\epsilon_{1j}, \ \epsilon_{2j}, \ \cdots, \ \epsilon_{j-1,j}, \ 0 \quad \textit{and} \quad \epsilon_{1j} - 1, \ \epsilon_{2j} - 1, \ \cdots, \ \epsilon_{j-1,j} - 1, \ 0$$

and call them respectively

$$P_1, \ P_2, \ \cdots, \ P_j \quad \textit{and} \quad Q_1, \ Q_2, \ \cdots, \ Q_j.$$

Construct the convex broken line $P_1 P_\sigma \cdots P_j$ above which all the remaining points P lie, and likewise construct the convex broken line $Q_1 Q_\theta \cdots Q_j$ below which all the remaining points Q lie. Let the acute angles which the successive sides of $P_1 P_\sigma \cdots P_j$ and $Q_1 Q_\theta \cdots Q_j$ make with the

axis of reals be $\phi_1, \phi_\sigma, \cdots$ *and* $\psi_1, \psi_\theta, \cdots$ *respectively. The critical rays between* $\arg x = 0$ *and* $\arg x = 2\pi$ *are then*

$$\frac{\pi}{2} - \phi_1, \ \frac{\pi}{2} - \phi_\sigma, \ \cdots, \ -\frac{\pi}{2} + \psi_\theta, \ -\frac{\pi}{2} + \psi_1,$$

in angular order. The asymptotic form of $g_{ij}(x)$ *changes uniformly from*

$$s_{i\alpha}(x) e^{2\pi\lambda_{\alpha j}\sqrt{-1}x} c_{\alpha j}^{(0)} \quad \text{to} \quad s_{i\beta}(x) e^{2\pi\lambda_{\beta j}\sqrt{-1}x} c_{\beta j}^{(0)}$$

along the critical ray $\arg x = \frac{1}{2}\pi - \phi_\alpha$, *and likewise from*

$$s_{i\gamma}(x) e^{2\pi(\lambda_{\gamma j}-1)\sqrt{-1}x} e^{-2\pi r_\delta \sqrt{-1}} c_{\gamma j}^{(\mu-1)} \quad \text{to} \quad s_{i\delta}(x) e^{-2\pi(r_{\delta j}-1)\sqrt{-1}x} e^{-2\pi r_\delta \sqrt{-1}} c_{\delta j}^{(\mu-1)}$$

along the critical ray $\arg x = -\frac{1}{2}\pi + \psi_\delta$. *Between the last critical ray of the first set and the first of the last set the asymptotic form is given by* $s_{ij}(x)$.

It is interesting to observe that with $g_{1j}(x), \cdots, g_{nj}(x)$ there are associated at most $j-1$ critical rays in the first and in the fourth quadrants so that, for example, when $j=1$ there are no critical rays. But we have $g_1(x) = u_1(x)$, $\cdots, g_n(x) = u_n(x)$ and this result is in accord with Theorem 1.

There are two special cases of exceptional interest. The first is that in which $\rho_1, \rho_2, \cdots, \rho_n$ are of equal absolute value. The quantities ϵ_{ij} are in this case all pure imaginaries and the broken lines $P_1, P_\sigma, \cdots$ and $Q_1, Q_\theta, \cdots$ reduce to segments of the imaginary axis, and we have $\phi_1 = \pi/2$, $\psi_1 = \pi/2$, so that the critical rays fall along the positive axis of reals, i. e., *do not exist* since this line is excluded throughout. In this event the elements of $G(x)$ preserve their asymptotic form throughout the entire plane, a result which agrees with that obtained in § 1. The second case is that in which the arguments of $\rho_1, \cdots, \rho_n$ are equal. In this case the quantities ϵ_{ij} are all real. In the first quadrant the only critical ray is the positive axis of imaginaries. In the fourth quadrant one finds critical rays different from the negative axis of imaginaries. How shall we reconcile these unsymmetric and apparently contradictory results, since by Theorem III the positive axis of imaginaries is not a critical ray? If one of the constants $c_{ij}^{(0)}$ which determine the change in asymptotic form *vanishes*, the formulas are still valid but do not give the dominant term.* The true critical rays in the first quadrant are symmetrically placed with respect to those in the fourth quadrant.

In general the effect of a zero value of $c_{ij}^{(0)}$ (or of $c_{ij}^{(\mu-1)}$) is to shift the actual change in asymptotic form to a *secondary* critical ray making a smaller angle with the positive axis of reals. The complete determination of its position may be effected when the constants $c_{ij}^{(k)}$ are known.

The critical rays for these solutions have been determined by Galbrun by

* Just as, for example, if $u = x^2$ we have $u \sim e^x(0)$ along the positive axis of reals, but this does not give the dominant term.

use of the Laplace transformation. It may be noted that the above results hold when the functions $a_{ij}(x)$ are only restricted as in (2), since only the form of the functions $p_{ij}(x)$ at $z = 0$ and $z = \infty$ has been made use of. An interesting difference between the case when the functions $a_{ij}(x)$ are rational and the case when they are not lies in the fact that there are a limited number of possible positions for the secondary rays in the first case, and an unlimited number in the second.

§ 7. The Fundamental Problem.

The properties which we have thus far adduced completely characterise the solutions of a linear difference system (1) with rational coefficients, as is evidenced by the following theorem:

Theorem VI. *Let $G(x)$ and $H(x)$ be two matrices of single-valued functions, analytic save for poles in the finite plane and such that their elements have the property*

$$\lim_{x=\infty} g_{ij}(x)x^{-\mu x}\rho_j^{-x}x^{-r_j} = s_{ij} \quad \text{and} \quad \lim_{x=\infty} h_{ij}(x)x^{-\mu x}\rho_j^{-x}x^{-r_j} = s_{ij}, \quad (|s_{ij}| \neq 0)$$

for x in any left or right half plane respectively, and furthermore let

$$G(x) = H(x)P(x)$$

where the elements of $P(x)$ are periodic of period 1. Then $G(x)$ and $H(x)$ are the first and second principal matrix solutions of a system (4) in which the elements of $A(x)$ are rational in x, at least if $\rho_1, \rho_2, \cdots, \rho_n$ are distinct.

Proof. If we put

$$A(x) = G(x+1)G^{-1}(x) = H(x+1)H^{-1}(x),$$

as we may by the hypothesis of the theorem, it is clear that the elements $a_{ij}(x)$ of $A(x)$ are single-valued and analytic in the finite plane except for poles. Furthermore, by direct computation, the elements $g'_{ij}(x)$ and $h'_{ij}(x)$ of $G^{-1}(x)$ and $H^{-1}(x)$ are seen to be such that

$$\lim_{x=\infty} g'_{ij}(x)x^{\mu x}(\rho_i e^{\mu})^x x^{r_i} = \sigma_{ij} \quad \text{and} \quad \lim_{x=\infty} h'_{ij}(x)x^{\delta\mu x}(\rho_i e^{-\mu})^{\delta x}x^{r_i} = \sigma_{ij},$$

in which σ_{ij} denotes as before the element in the ith row and jth column of the matrix inverse to (s_{ij}). The variable x lies in any left and right half plane respectively. Therefore we have

$$a_{ij}(x) = \sum_{\tau=1}^{n} g_{i\tau}(x+1)g'_{\tau j}(x) = \sum_{\tau=1}^{n} h_{i\tau}(x+1)h'_{\tau j}(x),$$

$$= \sum_{\tau=1}^{n} (x+1)^{\mu(x+1)}(\rho_\tau e^{-\mu})^{x+1}(x+1)^{r_\tau}(s_{i\tau}+\epsilon_{i\tau})x^{-\mu x}(\rho_\tau e^{-\mu})^{-x}x^{-r_\tau}(\sigma_{\tau j}+\epsilon'_{ij}),$$

where $\epsilon_{i\tau}$ and $\epsilon'_{\tau j}$ tend to zero *for all large values of* x, that is, in overlapping right and left half planes. This may be written

$$a_{ij}(x) = x^{\mu} e^{-\mu} \left(1 + \frac{1}{x}\right)^{\mu(x+1)} \sum_{\tau=1}^{n} (s_{i\tau} \rho_{\tau} \sigma_{\tau j} + \epsilon''_{i\tau j}),$$

where $\epsilon''_{i\tau j}$ tends to zero as x becomes infinite.

Thus $a_{ij}(x)$ must have a pole of order μ at $x = \infty$ with leading coefficient

$$a_{ij} = \sum_{\tau=1}^{n} s_{i\tau} \rho_{\tau} \sigma_{\tau j},$$

and is a rational function of x. In order to find the roots of the characteristic equation (3) in this case, we make an initial linear transformation with constant coefficients on $g_1(x), \cdots, g_n(x)$ in the linear difference system

$$g_i(x) = \sum_{i=1}^{n} a_{ij}(x) g_j(x)$$

which has been constructed. The constants s_{ij} undergo the same transformation and may be reduced to δ_{ij}; thus we shall have $a_{ij} = \delta_{ij} \rho_j$, so that the invariant characteristic equation is

$$(\rho - \rho_1)(\rho - \rho_2) \cdots (\rho - \rho_n) = 0$$

with distinct roots $\rho_1, \cdots, \rho_n$ not zero. Hence the above system satisfies the restrictions of § 1. According to the definition of principal matrix solutions, $G(x)$ and $H(x)$ must be the first and second principal matrix solutions, since the principal matrix solutions are the only ones which preserve a definite asymptotic form $s_{ij}(x)$ even to terms of the first order in any left or right half plane respectively.

It is evident that the $n^2 + 2n$ *exponential constants* ρ_i, r_i, s_{ij} together with the functions $p_{ij}(x)$ characterize the principal solutions more or less completely. Let us turn again to the case in which the coefficients $a_{ij}(x)$ are polynomials of degree μ in x. In this case the equations (51) show that the functions $p_{ij}(x)$ involves $n^2\mu - n$ further *polynomial constants* $c_{ij}^{(l)}$. But among the exponential constants there are n which are not essential since we can alter any set $s_{1k}, \cdots, s_{nk}$ by an arbitrary multiplier and not affect the associated linear difference system (1). Hence there have been specified in all

$$(n^2 + 2n) + (n^2\mu - n) - n \qquad \text{or} \qquad n^2(\mu + 1)$$

characteristic constants counting only the ratios $s_{1k}, s_{2k}, \cdots, s_{nk}$ as characteristic constants. This number is precisely the number of arbitrary coefficients in the n^2 polynomials $a_{ij}(x)$, each of degree μ or less in x. Now the system (1)

determines the constants, but not uniquely since $r_1, \cdots, r_n$ may be replaced by any determination $r_1 + 2m_1\sqrt{-1}, \cdots, r_n + 2m_n\sqrt{-1}$ ($m_1, \cdots, m_n$ integers) in the formal expansions. There is however exactly one set of characteristic constants for each such determination. If the characteristic constants are independent, one may conversely regard the coefficients in the polynomials $a_{ij}(x)$ as functions of the characteristic constants. The following definition and theorem go to show that conversely the characteristic constants do describe all the essential properties of the solutions:

Definition. Two linear systems (1) are said to belong to the same *class* if one may be transformed into the other by a linear transformation with rational coefficients.

THEOREM VII. *Two linear difference systems* (1) *in which* $\rho_1, \rho_2, \cdots, \rho_n$ *are distinct and in which the functions* $a_{ij}(x)$ *are polynomials in* x *of degree* μ *will belong to the same class if their characteristic constants are the same.*

Proof. Let

$$(54)\quad G_1(x+1) = A_1(x)G_1(x) \qquad \text{and} \qquad G_2(x+1) = A_2(x)G_2(x)$$

be two systems of this kind, expressed in matrix notation, which possess the same characteristic constants. For convenience we suppose both of the equations transformed into a normal form in which

$$(55)\qquad a_{ij}(x) = \rho_j(\delta_{ij}x^\mu + a_{ij}^{(1)}x^{\mu-1} + \cdots).$$

This may always be effected by one and the same linear transformations with constant coefficients,

$$G(x) = C\bar{G}(x),$$

namely by that transformation which reduces the constants s_{ij} to δ_{ij},

$$(\delta_{ij}) = (c_{ij})(s_{ij}).$$

This is always possible by (7), and if one substitutes in the new formal expansions, it appears at once that the transformed elements of $A_1(x)$ and $A_2(x)$ have the form (55). The exponential constants ρ_j, r_j and the polynomial constants are of course unaltered by this transformation, and we have explicitly in each case

$$(56)\qquad r_i = a_{ii}^{(1)}, \qquad (i = 1, 2, \cdots, n).$$

This simplification being effected, let us consider the relation between the first principal matrix solutions $G_1(x)$ and $G_2(x)$. Write

$$(57)\qquad G_1(x) = R(x)G_2(x).$$

We find then

$$(58)\quad R(x) = G_1(x)G_2^{-1}(x) = H_1(x)P(x)[H_2(x)P(x)]^{-1} = H_1(x)H_2^{-1}(x),$$

where $H_1(x)$ and $H_2(x)$ are the second principal matrix solutions and $P(x)$ is the common matrix of fundamental periodic functions. It is now easy to demonstrate that $R(x)$ is a matrix of rational functions of x. The elements $a_{ij}(x)$ of this matrix have only poles in the finite plane and the relation (58) shows that

$$\lim_{x=} R(x) = \delta_{ij} \tag{59}$$

in the complete vicinity of $x = \infty$ on account of the common asymptotic form of G_1 and G_2, H_1 and H_2. In consequence of (59) the elements $r_{ij}(x)$ must be rational functions of x, analytic at $x=\infty$ and taking on the value δ_{ij} at that point.

Substitute the value of $G_1(x)$ given in (57) in the first of the given difference systems. There results

$$G_2(x+1) = [R^{-1}(x+1)A_1(x)R(x)]G_2(x).$$

A comparison with the second of the given systems shows then that

$$A_1(x)R(x) = R(x+1)A_2(x). \tag{60}$$

This is the condition that the two systems belong to the same class and are transformable, the first into the second, by (57).

Even if this theorem is true there remains the possibility that not all of the characteristic constants are independent, although we are now certain that these constants effectively determine the transcendental properties of the solutions. We may however say something more of the dependence of solutions on the characteristic constants.

In the first place we observe that the exponential constants are expressed in terms of the coefficients in (56) when the reduction to normal form has been effected. Now the principal matrix solutions of (1) were constructed by means of certain limit determinants and a process of summation based on a contour integral. All the processes employed are uniformly convergent in the coefficients $a_{ij}^{(k)}$ and involve them analytically. *It follows that the two principal solutions* $G(x)$ *and* $H(x)$ *of* (1) *are analytic functions of the constants* $a_{ij}^{(k)}$ as well as of x, and hence the elements of

$$P(x) = G(x)H^{-1}(x)$$

have the same property. This necessitates, of course, that the coefficients $c_{ij}^{(l)}$ be analytic in the coefficients $a_{ij}^{(l)}$. Let us take $\rho_1, \cdots, \rho_n, r_1, \cdots, r_n$ as fixed, which amounts to choosing $a_{ii}^{(1)}$ in (56) once for all. The coefficients $c_{ij}^{(l)}$ are then single-valued analytic functions of the remaining coefficients $a_{ij}^{(l)}$, which appear in precisely equal numbers.

Now only two things are possible: either the characteristic constants are independent and one may conversely regard the quantities $a_{ij}^{(l)}$ as analytic functions

of the quantities $c_{ij}^{(l)}$, or else they are *dependent*. In this latter case we should be able to choose

$$a_{ij}^{(k)} = \phi_{ijk}(\lambda) \qquad \text{for all } i, j, k,$$

where $\phi_{ijk}(\lambda)$ is an analytic function of the parameter λ, in such a way that the characteristic constants are independent of λ. The functions $a_{ij}(x)$ are then dependent on λ, so that we may write $A(x) = A(x, \lambda)$. Furthermore we can take generally

$$A_1(x) = A(x, \lambda_0), \qquad A_2(x) = A(x, \lambda)$$

as two determinations of $A(x)$ which give equations (54) with the same characteristic constants, and we may take $\lambda = \lambda_0$ as a value of λ for which the functions $\phi_{ijk}(x)$ are analytic. The matrix $R(x)$ is defined by (57), where $G_1(x)$ and $G_2(x)$ denote again the first principal solutions in (54). The elements $r_{ij}(x)$ are rational functions of x by the preceding theorem.

We will now show that the functions $r_{ij}(x)$, which depend also on λ, must be polynomials in x. To this end we make the further assumption that no two of the μ^n roots of the determinant equation $|A(x, \lambda^0)| = 0$ differ by integers. This is manifestly the general case.

If one of the elements of R has a pole at a point, it will be possible to form a group of such points

$$\alpha, \alpha + 1, \cdots, \beta$$

containing the special point and such that $\alpha - 1$ and $\beta + 1$ are poles of no element of R. Now if we consider (60) at $x = \beta$, we see that the right-hand member is analytic at this point, and that the left-hand member will have at least one element with a pole at $x = \beta$ unless $|A_1(x)| = 0$ at $x = \beta$. Hence we must have $A_1(\beta) = 0$. Likewise if we consider the same equation at $x = \alpha - 1$, we see that the left-hand member is analytic at this point, and that the right-hand member will have at least one element with a pole unless $|A_2(x)| = 0$ at $x = \alpha - 1$. Hence we must have $A_2(\alpha - 1) = 0$. But for λ near to λ_0, the roots of

$$|A_1(x)| = |A(x, \lambda_0)| = 0 \qquad \text{and} \qquad |A_2(x)| = A(x, \lambda)| = 0$$

will nearly coincide and no two of them, as $\alpha - 1$ and β respectively, can be congruent and distinct on account of the hypothesis concerning the roots of $|A(x, \lambda_0)| = 0$. It follows that the elements of $R(x)$ can have no poles in the finite plane, and hence are all polynomials in x. According to (59) these polynomials reduce to δ_{ij} at $x = \infty$, and hence are identically equal to δ_{ij}. *The elements of* $A(x, \lambda)$ *will not depend on the parameter* λ, *contrary to hypothesis.*

Hence the characteristic constants are independent, and the coefficients in the polynomials $a_{ij}(x)$ may be conversely regarded as analytic functions of these characteristic constants. A fundamental problem of the linear difference system is to study this functional relation and in particular *to determine whether there exists a system* (1) *with coefficients polynomial in* x *of degree not greater than* μ *having any assigned set of characteristic constants.*

PRINCETON UNIVERSITY.